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A KEY
TO
AN INTRODUCTORY COURSE OF
Plane Trigonometry and Logarithms.
(THIRD EDITION.)

BY
JOHN WALMSLEY, B.A.

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P R E F A C E.

THE present volume has been prepared in compliance with the expressed wishes of several mathematical teachers.

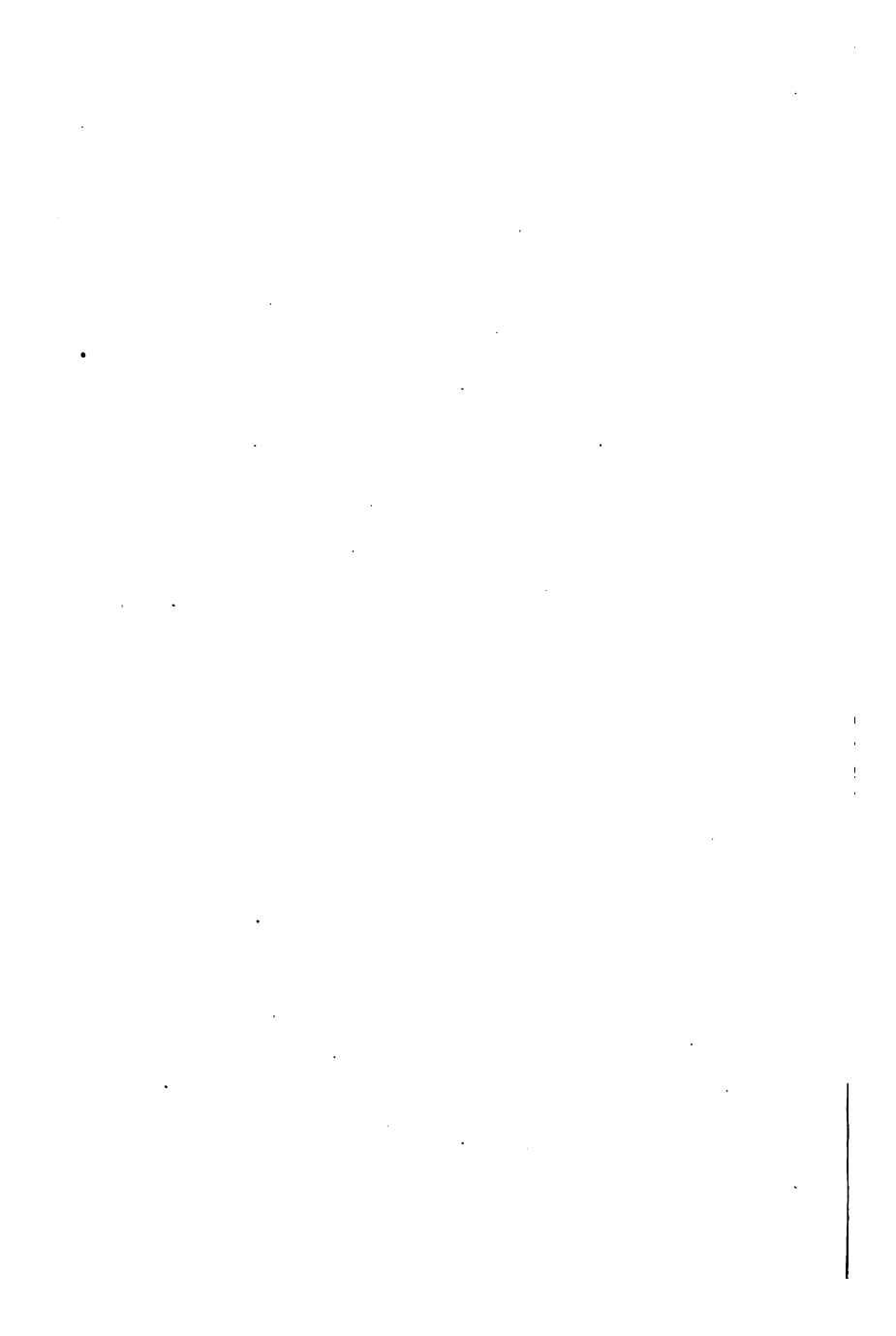
Judiciously employed, it is hoped that it will be found of service to private students also.

Of necessity, the solutions herein contained are not, in many cases, "set out" as the ordinary work of a student should be; but this can hardly be an inconvenience to those for whom the book is intended, especially when the number and variety of the examples worked out in the text-book are considered.

The author cannot sufficiently thank those gentlemen who have so generously and well assisted him in revising for the press—a labour, in such cases as the present, arduous in no ordinary degree. He trusts that any errors which may be found will be such as are evident at sight, and therefore cause no inconvenience.

J. W.

ECCLES, MANCHESTER.



SOLUTIONS.



[Observe that, in these pages,—(1) In the case of identities, f represents the first side given, and l the other side. (2) In the case of equations, each solution begins with the first step after the statement of the equation.]

EXERCISE I.

1. $\frac{7}{8} = \frac{1}{2}$; $\frac{1}{2} \frac{2}{3} = \frac{1}{3}$; $1 + \frac{1}{2} = 1 \times 2 = 2$; $\frac{1}{2} + \frac{2}{3} = \frac{1}{2} \times \frac{2}{3} = \frac{2}{3}$;
 $\frac{2}{3} + 3\frac{1}{3} = \frac{2}{3} \times \frac{1}{3} = \frac{1}{9}$. 2. 1 ft. : 1 yd. = 1 ft. : 3 ft. = 1 : 3;
 1 yd. : 10 $\frac{1}{2}$ in. = 36 : $\frac{21}{2}$ = 72 : 21 = 24 : 7; 6 f. : 1 $\frac{1}{2}$ m. = $\frac{2}{3}$ m. : $\frac{1}{3}$ m.
 = 9 : 16. 3. 1 ft. : 1 yd. = 1 ft. : 3 ft. = 1 : 3,
 1 sq. ft. : 1 sq. yd. = 1 sq. ft. : 9 sq. ft. = 1 : 9, and $\frac{1}{3} = 3 \times \frac{1}{9}$.
 4. 1 ft. : 1 in. = 12 in. : 1 in. = 12, 1 sq. ft. : 3 \times 4 sq. in. = 144 sq. in. :
 12 sq. in. = 12; ratio equal. 5. $\frac{x \text{ hf.-cra.}}{x \text{ hf.-grains.}} = \frac{5x}{21x} = \frac{5}{21}$.
 6. $x : 5 \text{ cwt.} = 3 : 7$, $\therefore x = \frac{15 \text{ cwt.}}{7} = 2 \text{ cwt. } 16 \text{ lbs.}$
 7. Let x' = unit, then $270 : 18x = 3 : 2$, $\therefore x = 10'$.

EXERCISE II.

1. $\frac{1}{2}$ of $90^\circ = 30^\circ$; $\frac{2}{3}$ of $90^\circ = 67^\circ 30'$; $\frac{1}{11}$ of $90^\circ = 139^\circ 5' 27'' \cdot \dot{2}7$.
 2. $\frac{1}{2}$ of $100^\circ = 50^\circ$; $\frac{2}{3}$ of $100^\circ = 133^\circ 33' 33'' \cdot \dot{3}$. $\frac{2}{3}$ of $100^\circ = 85^\circ 77' 42'' \cdot \dot{86}$.
 3. $\begin{array}{r} .051544\dot{3} \text{ rt. } \angle \\ \underline{90} \\ 4^\circ 638990 \\ \underline{60} \\ 38' 3394 \\ \underline{60} \\ 20'' 364 \end{array}$ $\begin{array}{r} 60 \mid 15'' \\ 60 \mid 41 \cdot 25 \\ 90 \mid 41 \cdot 6875 \\ \hline .463194 \text{ rt. } \angle \end{array}$

4. $81^\circ = \frac{1}{2}^\circ \times 81^\circ = 90^\circ$. $33^\circ.75 = \frac{1}{2}^\circ \times 33^\circ.75 = 37^\circ 50'$.
 $150^\circ = \frac{1}{2}^\circ \times 150^\circ = 166^\circ 66' 66'' \cdot 6$. $74^\circ 31' 52'' \cdot 5 = 74^\circ 53' 125 =$
 $\frac{1}{2}^\circ \times 74^\circ 53' 125 = 82^\circ 81' 25''$. 5. $150^\circ = \frac{1}{2}^\circ \times 150^\circ = 135^\circ$.
 $18^\circ 75' = \frac{1}{2}^\circ \times 18^\circ.75 = 16^\circ.875 = 16^\circ 52' 30''$. $63^\circ 25' 36'' \cdot 45 =$
 $\frac{1}{2}^\circ \times 63^\circ.253645 = 56^\circ.9282805 = 56^\circ 55' 41'' \cdot 8$.

6.
$$\begin{array}{r} 60 \quad | \quad 48'' \cdot 3 \\ 60 \quad | \quad 5' \cdot 805 \\ 90 \quad | \quad 72^\circ \cdot 09675 \end{array}$$

$$\begin{array}{r} 60 \quad | \quad 1'' \\ 60 \quad | \quad 1 \cdot 0167 \quad \&c. \end{array}$$

7.
$$\begin{array}{r} \cdot 801075 \text{ rt. } \angle = 80^\circ 10' 75'' \\ 90 \\ 13 \cdot 77045 \text{ deg.} \\ 60 \\ 46 \cdot 227 \text{ min.} \\ 60 \\ 13 \cdot 62 \text{ sec.} \\ 13^\circ 46' 13'' \cdot 62 \end{array}$$

$$\begin{array}{r} \cdot 0101011 \text{ rt. } \angle \\ 90 \\ 0^\circ \cdot 909099 \\ 60 \\ 54' \cdot 54594 \\ 60 \\ 32'' \cdot 7564 \end{array}$$

$$\begin{array}{r} \cdot 0404044 \text{ rt. } \angle \\ 90 \\ 3^\circ 636396 \\ 60 \\ 38' 18376 \\ 60 \\ 11'' \cdot 0256 \end{array}$$

CHAPTER II.

1. $\frac{1}{2}$ right angle $= \frac{1}{2} \times \frac{1}{2}\pi = \frac{1}{4} \times 3 \cdot 1416 = \cdot 7854$. $\frac{1}{2}$ rt. $\angle = \frac{1}{2} \times \frac{1}{2}\pi =$
 $\frac{1}{2} \times 6 \cdot 2832 = 2 \cdot 0944$. $\frac{1}{2}$ rt. $\angle = \frac{1}{2} \times \frac{1}{2}\pi = \frac{1}{4} \times 9 \cdot 4248 = 1 \cdot 3464$.

2. $30^\circ = \frac{1}{2}$ of $90^\circ = \frac{1}{2} \times \frac{1}{2}\pi = \frac{1}{4}\pi = \cdot 5236$. $22^\circ 30' = \frac{1}{2} \times \frac{1}{2}\pi = \frac{1}{4}\pi = \cdot 3927$.

$90 : 156\frac{1}{2}^\circ = \frac{\pi}{2} : \frac{3 \cdot 1416 \times 627}{2 \times 4 \times 90} = 2 \cdot 7358$.

$90 : 96^\circ 12' 12'' = \frac{\pi}{2} : \frac{3 \cdot 1416 \times 28861}{2 \times 90 \times 300} = 1 \cdot 679069$.

3. $150^\circ = \frac{3}{2} \cdot \frac{\pi}{2} = \frac{3 \times 3 \cdot 1416}{4} = 2 \cdot 3562$.

$100^\circ : 73^\circ 62' 3'' \cdot 7 = \frac{\pi}{2} : \frac{3 \cdot 1416 \times 73 \cdot 62037}{2 \times 100} = 1 \cdot 1564288$.

4. $\frac{5}{8}\pi = \frac{5}{8}$ of $90^\circ = 150^\circ$. $\frac{1}{2}\pi : 5323 = 90^\circ : \frac{90^\circ \times 5323}{1 \cdot 5708} =$

$30^\circ 30'$ nearly. $\frac{1}{2}\pi : 8447 = 90^\circ : \frac{90^\circ \times 8447}{1 \cdot 5708} = 48^\circ 23' 51'' \cdot 5$.

5. $\frac{1}{2}\pi = \frac{1}{2} \times 200^\circ = 40^\circ$.

$\frac{1}{2} : 1 \cdot 7482 = 100 : \frac{100^\circ \times 1 \cdot 7482}{1 \cdot 5708} = 111^\circ 29' 36''$.

6. $150^\circ = \frac{1}{2}^\circ \times 150^\circ = \frac{1}{2} \times 500^\circ = 166^\circ 66' 66'' \cdot 7$.

$18^\circ 7' 14'' \cdot 16 = 18 \cdot 1206 = \frac{1}{2}$ of $18^\circ \cdot 1206 = 20^\circ 13' 40''$.

7. $125^\circ = \frac{5}{16}$ of $125^\circ = \frac{1}{4} \times 225^\circ = 112^\circ 30'$.
 $56^\circ 24' 25'' = 56^\circ \cdot 2425 = \frac{1}{10} \times 9 \times 56 \cdot 2425 = 50^\circ \cdot 61825 = 50^\circ 37' 5'' \cdot 7$.
8. Reqd. c. meas. $= \frac{4 \text{ ft. } 9 \text{ in.}}{7 \text{ ft. } 11 \text{ in.}} = \frac{57}{95} = \frac{3}{5} = \cdot 6$.
9. $1 \cdot 7 = \frac{11 \text{ yds. } 1 \text{ ft.}}{\text{radius}}$, $\therefore \text{rad.} = \frac{34 \text{ ft.}}{1 \cdot 7} = 20 \text{ feet.}$
10. $\frac{\text{required arc}}{\text{radius}} = \text{c. m. of } \angle = \frac{3}{4}$; $\therefore \text{arc} = \frac{3}{4} \times 204 \text{ m.} = 153 \text{ m.}$
11. $\frac{\text{required arc}}{21} = \text{c. meas. of } 30^\circ = \frac{\pi}{6} = \frac{22}{6 \cdot 7} = \frac{11}{21}$, $\therefore \text{arc} = 11$.
12. Apply Euc. I. 32 and Cor. 1. The angle in case of—
 (1) triangle $= \frac{1}{3}$ of $180^\circ = 60^\circ$ or $\frac{1}{3}\pi$.
 (2) hexagon $= \frac{1}{6}$ of $(12 \times 90^\circ - 4 \times 90^\circ) = \frac{1}{6}$ of $8 \times 90^\circ = 120^\circ$ or $\frac{2}{3}\pi$.
 (3) decagon $= \frac{1}{10}$ of $(20 \times 90^\circ - 4 \times 90^\circ) = \frac{1}{10}$ of $16 \times 90^\circ = 144^\circ$ or $\frac{4}{5}\pi$.
 (4) quindecagon $= \frac{1}{15}$ of $(30 \times 90^\circ - 4 \times 90^\circ) = \frac{2}{15}$ of $90^\circ = 12^\circ$ or $\frac{2}{15}\pi$.
13. $\frac{1}{2}$ of $90 : 60 = 1 : 60 \times \frac{1}{60} = 2 \cdot 6$. $\frac{1}{2}$ of $100 : 60 = 1 : 60 \times \frac{1}{60} = 2 \cdot 4$.
14. $\frac{1}{4}\pi : \cdot 5236 = 1 : \frac{5236 \times 4}{3 \cdot 1416} = \frac{2}{3} = \cdot 6$.

EXERCISE III.

2. $c^2 = 9^2 + 12^2 = 81 + 144 = 225$, $\therefore c = 15$.
3. $a^2 = c^2 - b^2 = (25 + 20)(25 - 20) = 45 \times 5 = 225$, $\therefore a = 15$.
4. $c^2 = 100 + 576 = 676$, $\therefore c = 26$.
5. $c^2 = 49 + 81 = 130$, $\therefore c = \sqrt{130} = \&c$.
6. $a^2 = (10 + 9)(10 - 9) = 19$, $\therefore a = 4 \cdot 36 \dots$
7. $b^2 = (10 + 1)(10 - 1) = 99$, $\therefore b = \&c$.
8. sq. on diagonal $= (144 + 25) \text{ yds.} = 169 \text{ yds.}$, $\therefore \text{diag.} = 13 \text{ yds.}$
9. sq. on diagonal $= 2 \cdot 20^2 \text{ sq. ft.}$, $\therefore \text{diag.} = 20\sqrt{2} = \&c$.
10. Let $x \text{ ft.}$ be side; then $2x^2 = (5\sqrt{2})^2 = 50$, $\therefore x = 5$.
11. sq. on breadth $= (30 + 27)(30 - 27) \text{ sq. yds.} = 57 \cdot 3 \text{ sq. yds.} = 171 \text{ sq. yds.}$ $\therefore \text{breadth} = \sqrt{171} \text{ yds.} = 13 \cdot 1 \dots \text{ yds.}$
12. Let $x \text{ in.}$ be side, $\therefore x^2 (\frac{1}{2}x)^2 = 10^2$, $\therefore x^2 = \frac{1}{3} \times 20^2$,
 $\therefore x = \frac{1}{3} \times 20\sqrt{3} = \&c$.
13. $BC^2 = (50 + 30)(50 - 30) \text{ sq. yds.} = 1600 \text{ sq. yds.}$, $\therefore BC = 40$.
14. sq. on dist. reqd. $= (125 + 75)(125 - 75) \text{ sq. yds.} = 10000 \text{ sq. yds.}$
 $\therefore \text{dist.} = 100 \text{ yds.}$
15. sq. on reqd. dist. $= (4 + \frac{4}{9}) \text{ sq. m.} = \frac{16}{9} \text{ sq. m.}$, $\therefore \text{dist.} = 3\frac{1}{3} \text{ miles.}$
16. sq. on width $= (75 + 60)(75 - 60) \text{ sq. ft.} = 135 \cdot 15 \text{ sq. ft.} = 9 \cdot 15^2 \text{ sq. ft.}$
 $\therefore \text{width} = 3 \cdot 15 \text{ ft.} = 45 \text{ feet.}$

17. sq. on dist. = $[(3.36)^2 + (1.4)^2]$ sq. m. = 13.2496 sq. m.,
 \therefore dist. = 3.64 m.

18. Noting the tacks made by sailing vessel, square on distance in miles = $30^2 + 40^2 = 2500$, \therefore dist. = 50 miles.
 \therefore time of steamer = $\frac{5}{11}$ hrs. = 4 hrs. 10 min.

EXERCISE IV.

$$(3.) \text{ (i.) } l = c \times \frac{a}{c} = a. \quad \text{(ii.) } l = ac \times \frac{b}{c} \times \frac{b}{a} = b^2.$$

$$\text{(iii.) } l = ab^2 \times \frac{c}{b} \times \frac{c}{a} \times \frac{c}{b} = c^3.$$

$$\text{(iv.) } f = \left(\frac{a}{c} + \frac{b}{c} \right) + \frac{b}{c} = \frac{a+b}{c} \times \frac{c}{b} = \frac{a+b}{b}.$$

$$\text{(v.) } f = \left(\frac{a}{c} + \frac{b}{c} + 1 \right) + \frac{a}{c} = \frac{a+b+c}{c} \times \frac{c}{a} = l.$$

$$\text{(vi.) } f = \frac{a^2}{b^2} - \frac{b^2}{a^2} = \frac{a^4 - b^4}{a^2 b^2} = \frac{(a^2 + b^2)(a^2 - b^2)}{a^2 b^2} = l.$$

CHAPTER III.

1. $90^\circ - 12^\circ = 78^\circ$; $90^\circ - 25^\circ 4' = 64^\circ 56'$; $90^\circ - 56^\circ 50' 44'' = 33^\circ 9' 16''$.
 $90^\circ - 88^\circ 58' 58'' = 1^\circ 1' 1'' = 11$.

2. $90^\circ - (45^\circ \mp A) = 90^\circ - 45^\circ \pm A = 45^\circ \pm A$.

3. $\tan A = \tan (90^\circ - A)$, $\therefore A = 90^\circ - A$, $\therefore A = 45^\circ$.

4. $\sin 5A = \sin (90^\circ - A)$, $\therefore 5A = 90^\circ$, $\therefore A = 18^\circ$.

5. $\sec 3A = \sec (90^\circ - 2A)$, $\therefore 3A = 90^\circ$, $\therefore A = 18^\circ$.

6. With the construction and diagram of Art. 21,

$$\sec 30^\circ = \frac{AB}{AD} = \frac{2a}{a\sqrt{3}} = \frac{2}{\sqrt{3}}; \quad \operatorname{cosec} 30^\circ = \frac{2a}{a} = 2;$$

$$\cot 30^\circ = \frac{a\sqrt{3}}{a} = \sqrt{3}; \quad \operatorname{vers} 30^\circ = 1 - \cos 30^\circ = 1 - \frac{a\sqrt{3}}{2a} = \frac{1}{2}(2 - \sqrt{3}).$$

With construction of Art. 20,

$$\sec 45^\circ = \frac{AB}{AC} = \frac{a\sqrt{2}}{a} = \sqrt{2}; \quad \operatorname{cosec} 45^\circ = \frac{a\sqrt{2}}{a} = \sqrt{2};$$

$$\cot 45^\circ = \frac{a}{a} = 1; \quad \operatorname{vers} 45^\circ = 1 - \frac{a}{a\sqrt{2}} = \frac{\sqrt{2}-1}{\sqrt{2}}.$$

With construction of Art. 21,

$$\sec 60^\circ = \frac{2a}{a} = 2, \quad \operatorname{cosec} 60^\circ = \frac{2a}{a\sqrt{3}} = \frac{2}{\sqrt{3}};$$

$$\cot 60^\circ = \frac{a\sqrt{3}}{a} = \sqrt{3}, \quad \operatorname{vers} 60^\circ = 1 - \frac{a}{2a} = \frac{1}{2}.$$

$$7. \sin^2 30^\circ : \sin^2 45^\circ : \sin^2 60^\circ = \frac{1}{4} : \frac{1}{2} : \frac{3}{4} = 1 : 2 : 3.$$

$$8. f = \frac{1 - \frac{1}{3}}{1 + \frac{1}{3}} = \frac{1}{2} = l. \quad 9. l = \frac{2 \times \frac{1}{\sqrt{3}}}{1 - \frac{1}{3}} = \frac{2}{\sqrt{3}} \times \frac{3}{2} = \frac{1}{\sqrt{3}} = f.$$

$$10. f = 4 \times \frac{1}{3} - 3 \times \frac{1}{3} = \frac{1}{3} - \frac{2}{3} = -\frac{1}{3}.$$

$$11. f = \frac{3 \times \frac{1}{\sqrt{3}} - \frac{1}{3\sqrt{3}}}{1 - \frac{1}{3}} = \frac{\sqrt{3} - \frac{1}{3\sqrt{3}}}{0} = l.$$

$$12. f = \frac{\frac{1}{\sqrt{2}} - \frac{1}{2}}{\frac{1}{\sqrt{2}} + \frac{1}{2}} = \frac{\sqrt{2} - 1}{\sqrt{2} + 1} = l. \quad 13. f = \frac{\frac{\sqrt{3}}{2} + \frac{1}{2}}{\frac{\sqrt{3}}{2} - \frac{1}{2}} = \frac{\sqrt{3} + 1}{\sqrt{3} - 1} = l.$$

$$14. f = 3 \times \frac{1}{3} - \frac{1}{3} = 1 - \frac{1}{3} = \frac{2}{3} = l.$$

$$15. l = \frac{\frac{1}{2}}{1} + \frac{1}{2} = 1, \text{ and } f = \frac{3}{2} - \frac{1}{2} = 1; \therefore f = l.$$

16. Make a rt. \angle d $\triangle ABC$ having a rt. $\angle C$ and $CA = 4$, $CB = 3$.
Then $AB^2 = 25$; $\therefore AB = 5$; $\therefore \sin A = \frac{BC}{AB} = \frac{3}{5}$; $\cos A = \frac{4}{5}$
 $\cot A = \frac{4}{3}$, $\sec A = \frac{5}{4}$, $\operatorname{cosec} A = \frac{5}{3}$, $\operatorname{vers} A = 1 - \frac{3}{5} = \frac{2}{5}$.

17. Make rt. \angle d $\triangle ABC$, with C right, $CB = 5$, $AB = 9$; then (Euc. 1. 47) $AC = 2\sqrt{14}$, $\therefore \tan A = \frac{5}{2\sqrt{14}}$; $\cos A = \frac{2\sqrt{14}}{9}$;
 $\operatorname{vers} A = 1 - \frac{2\sqrt{14}}{9} = \frac{1}{9}(9 - 2\sqrt{14})$.

18. Make rt. \angle d \triangle with C rt., $CA = 1$, $AB = 2\sqrt{2}$; then $CB = \sqrt{7}$.
 $\therefore \sin A = \frac{\sqrt{7}}{2\sqrt{2}} = \frac{1}{4}\sqrt{14}$, $\cot A = \frac{1}{\sqrt{7}}$.

19. Make rt. \angle d \triangle with C rt., $AB = 1$, $CA = c$, $\therefore CB = \sqrt{1 - c^2}$.
 $\therefore \sin A = \frac{CB}{AB} = \frac{\sqrt{1 - c^2}}{1} = \sqrt{1 - c^2}$; $\tan A = \frac{\sqrt{1 - c^2}}{c}$, &c.

20. Let $\cot A = c = \frac{c}{1}$; and make rt. \angle d \triangle with C rt., $CA = c$,
 $CB = 1$, $\therefore AB = \sqrt{1 + c^2}$; $\therefore \sin A = \frac{1}{\sqrt{1 + c^2}} = \frac{1}{\sqrt{(1 + \cot^2 A)}}$,
 $\cos A = \frac{c}{\sqrt{(1 + c^2)}} = \&c.$

21. $\sec A = \frac{4}{3}$. Make ABC with C rt.; also $CA = 9$, $AB = 4$. This construction will fail.

22. Construct ABC with C rt.; and make $AB = p$, $CB = \sqrt{p^2 - q^2}$,
 $\therefore CA = q$, $\therefore \sin A = \frac{\sqrt{p^2 - q^2}}{p}$, $\cos A = \frac{q}{p}$, $\tan A = \frac{\sqrt{p^2 - q^2}}{q}$, &c.

CHAPTER IV.

1. $l = \cos A \frac{\sin A}{\cos A} = f.$ $l = \sin A \frac{\cos A}{\sin A} = f.$
2. $f = \tan^2 A \frac{1}{\tan A} = l.$ $l = \frac{1}{\cos A} \times \sin A = f.$
3. $l = \sqrt{\left(\frac{1}{\cos^2 A} \sin^2 A + 1\right)} = \sqrt{(\tan^2 A + 1)} = f.$
4. $l = \frac{1}{\sin A} \cos A = f.$ 5. $f = \frac{\sin^4 \theta}{\cos^2 \theta} \times \frac{1}{\cos^2 \theta} \times \frac{\cos^6 \theta}{\sin^6 \theta} = \frac{\cos^2 \theta}{\sin^2 \theta},$
and $l = \frac{1}{\cos^2 \theta} \times \frac{\cos^3 \theta}{\sin^3 \theta} \times \frac{\sin^3 \theta}{1} \times \frac{\cos \theta}{\sin \theta} = \frac{\cos^2 \theta}{\sin^2 \theta}.$
6. $f = \cos(90^\circ - A) \times \frac{1}{\sin(90^\circ - A)} = l.$
7. $l = \sec A \times \frac{1}{\operatorname{cosec} A} = \frac{1}{\cos A} \times \frac{\sin A}{1} = f.$
8. $l = \sin^2 \theta + \cos^2 \theta + 2 \sin \theta \cos \theta = f.$
9. $f = \sin \theta \times \sin \theta + \cos \theta \times \cos \theta = l.$
10. $f = \frac{1}{\cos^2 \theta} + \frac{1}{\sin^2 \theta} = \frac{\sin^2 \theta + \cos^2 \theta}{\cos^2 \theta \sin^2 \theta} = \frac{1}{\cos^2 \theta \sin^2 \theta} = l.$
11. $f = \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} = \frac{\sin^2 \theta + \cos^2 \theta}{\cos \theta \sin \theta} = \frac{1}{\cos \theta \sin \theta} = l.$
12. $f = (\cos^2 \theta + \sin^2 \theta)(\cos^2 \theta - \sin^2 \theta) = l.$
13. $f = \left(\cos \theta \times \frac{\sin \theta}{\cos \theta}\right)^2 + \left(\sin \theta \times \frac{\cos \theta}{\sin \theta}\right)^2 = \sin^2 \theta + \cos^2 \theta = 1.$
14. $f = \left(\frac{\sin A}{\cos A} \times \cos A\right)^2 + \left(\frac{\cos A}{\sin A} \times \sin A\right)^2 = \sin^2 A + \cos^2 A = 1.$
15. $f = \left(\frac{1}{\cos \theta} \frac{\cos \theta}{\sin \theta}\right)^2 + \left(\frac{1}{\sin \theta} \frac{\sin \theta}{\cos \theta}\right)^2 = \frac{1}{\sin^2 \theta} + \frac{1}{\cos^2 \theta} = \frac{\sin^2 \theta + \cos^2 \theta}{\sin^2 \theta \cos^2 \theta} = l.$
16. $f = \sin \theta \left(1 - \frac{1}{\sin \theta}\right) = \sin \theta - 1 = l.$
17. $f = \sin A + \cos A \frac{\cos A}{\sin A} + \sin A \frac{\sin A}{\cos A} + \cos A$
 $= \frac{\sin^2 A + \cos^2 A}{\sin A} + \frac{\sin^2 A + \cos^2 A}{\cos A} = l.$

$$18. f = \frac{\operatorname{cosec}^2 A + \sec^2 A + 2 \operatorname{cosec} A \sec A}{\operatorname{cosec}^2 A + \sec^2 A} \times \frac{\sin^2 A \cos^2 A}{\cos^2 A \cos^2 A}$$

$$= \frac{\cos^2 A + \sin^2 A + 2 \sin A \cos A}{\sin^2 A + \cos^2 A} = 1.$$

$$19. f = \frac{\sec A \cot A - \operatorname{cosec} A \tan A}{\cos A - \sin A} \times \frac{\sin A \cos A}{\sin A \cos A}$$

$$= \frac{\cos A - \sin A}{\cos A - \sin A} \times \frac{1}{\sin A \cos A} = 1.$$

$$20. f = \sin^2 A (1 - \sin^2 B) - (1 - \sin^2 A) \sin^2 B = 1.$$

$$21. f = \cos^2 A (1 - \sin^2 B) - (1 - \cos^2 A) \sin^2 B = 1.$$

$$22. \cos A = \frac{\cos A}{\sin A} \sin A = \cot A \frac{1}{\operatorname{cosec} A} = \frac{\sqrt{(\operatorname{cosec}^2 A - 1)}}{\operatorname{cosec} A}.$$

$$\operatorname{cosec} A = \frac{1}{\sin A} \frac{\cos A}{\cos A} = \cot A \sec A = \frac{\sec A}{\tan A} = \frac{\sqrt{(\tan^2 A + 1)}}{\tan A}.$$

$$23. \tan A = \frac{\sqrt{(1 - \cos^2 A)}}{\cos A}; \quad \cot A = \frac{1}{\tan A} = \frac{1}{\sqrt{(\sec^2 A - 1)}}.$$

$$24. \sin \theta = \frac{\sin \theta}{\cos \theta} \cos \theta = \frac{\tan \theta}{\sec \theta} = \frac{\tan \theta}{\sqrt{(\tan^2 \theta + 1)}}.$$

$$\operatorname{cosec}^2 A = \frac{1}{\sin^2 A} = \frac{1}{1 - \cos^2 A} = \frac{1}{(1 + \cos A)(1 - \cos A)} = \frac{1}{(2 - \operatorname{vers} A) \operatorname{vers} A}.$$

$$25. \sin A = \sqrt{(1 - \cos^2 A)} = \sqrt{(1 - .36)} = \sqrt{.64} = .8;$$

$$\tan A = \frac{.8}{.6} = 1.\bar{3}; \quad \sec A = \frac{1}{.6} = 1.\bar{6};$$

$$\operatorname{cosec} A = \frac{1}{.8} = 1.25; \quad \cot A = \frac{.6}{.8} = .75, \text{ \&c.}$$

$$26. \cos A = \sqrt{(1 - \frac{1}{2} \frac{1}{2})} = \sqrt{\frac{3}{4}} = \frac{1}{2} \sqrt{26}. \quad \tan A = \frac{11}{15} \times \frac{15}{2 \sqrt{26}} = \frac{11}{2 \sqrt{26}} \sqrt{26}.$$

$$\sec A = \frac{1}{\cos A} = \frac{15}{2 \sqrt{26}} = \frac{15}{52} \sqrt{26}, \text{ \&c.}$$

$$27. \tan A = \frac{c}{a}, \quad \sec A = \sqrt{\left(\frac{c^2}{a^2} + 1\right)} = \frac{\sqrt{(c^2 + a^2)}}{a}.$$

$$\operatorname{cosec} A = \sqrt{\left(\frac{a^2}{c^2} + 1\right)} = \frac{\sqrt{(a^2 + c^2)}}{c}.$$

$$\sin A = \frac{c}{\sqrt{(a^2 + c^2)}}, \quad \cos A = \frac{a}{\sqrt{(a^2 + c^2)}}.$$

$$28. 2 \sin^2 \theta = 1, \quad \therefore \sin \theta = \frac{1}{\sqrt{2}}, \quad \therefore \theta = \frac{1}{4}\pi.$$

$$29. 2 \tan^2 \theta = \tan^2 \theta + 1, \quad \therefore \tan^2 \theta = 1, \quad \therefore \theta = \frac{1}{4}\pi.$$

30. $2 \cos^2 \theta = 1\frac{1}{2}$, $\cos \theta = \frac{1}{2}\sqrt{3}$, $\therefore \theta = \frac{1}{3}\pi$.
31. $2 \cos^2 \theta - 3 \cos \theta + 1 = 0$, $\therefore \cos \theta = \frac{1}{2}(3 \pm \sqrt{1}) = 1$ or $\frac{1}{2}$, $\therefore \theta = \frac{1}{3}\pi$.
32. $1 + \cot^2 \theta = 2 \sec \theta \cot \theta = 2 \operatorname{cosec} \theta$, $\therefore \operatorname{cosec} \theta = 2$, $\therefore \theta = \frac{1}{2}\pi$.
33. $2 - 2 \cos^2 \theta + 5 \cos \theta = 4$, $\therefore 2 \cos^2 \theta - 5 \cos \theta + 2 = 0$,
 $\therefore \cos \theta = \frac{1}{2}(5 \pm \sqrt{9}) = 2$ or $\frac{1}{2}$, $\therefore \theta = \frac{1}{3}\pi$.
34. $\tan^2 \theta + \tan \theta - 2 = 0$, $\therefore \tan \theta = \frac{1}{2}(-1 \pm \sqrt{9}) = 1$ or -2 , $\therefore \theta = \frac{1}{4}\pi$.
35. $4 \sin^4 \theta - 7 \sin^2 \theta + 3 = 0$, $\therefore \sin^2 \theta = \frac{1}{8}(7 \pm \sqrt{1}) = 1$ or $\frac{3}{4}$,
 $\therefore \sin \theta = \frac{1}{2}\sqrt{3}$, $\therefore \theta = \frac{1}{3}\pi$.
36. $4 \cos^2 \theta - 3 \cos \theta + \frac{1}{2} = 0$, $\therefore \cos \theta = \frac{1}{2}(3 \pm \sqrt{1}) = \frac{3}{2}$ or $\frac{1}{2}$, $\therefore \theta = \frac{1}{3}\pi$.
37. $2 \operatorname{cosec}^2 \theta - 5 \operatorname{cosec} \theta + 2 = 0$, $\therefore \operatorname{cosec} \theta = \frac{1}{2}(5 \pm \sqrt{9}) = 2$ or $\frac{1}{2}$, $\therefore \theta = \frac{1}{2}\pi$.
38. $4 \tan^2 A = \cot A \tan A = 1$, $\therefore \tan A = \pm \frac{1}{2}$.
39. $25 - 25 \cos^2 A = 16$, $\therefore \cos^2 A = \frac{9}{25}$, $\therefore \cos A = \pm \frac{3}{5}$.
40. $3 \sin^2 A + 2 \sin A - 1 = 0$, $\therefore \sin A = \frac{1}{3}(-2 \pm \sqrt{16}) = \frac{1}{3}$ or -1 .
41. $\tan^2 A - 4 \tan A + 1 = 0$, $\therefore \tan A = \frac{1}{2}(4 \pm \sqrt{12}) = 2 \pm \sqrt{3}$.
42. $m^2 \sin^2 A = n^2(1 - \sin^2 A)$, $\therefore \sin^2 A(m^2 + n^2) = n^2$,
 $\therefore \sin A = \pm \frac{n}{\sqrt{m^2 + n^2}}$.
43. $\frac{\sin \theta}{\cos^2 \theta} = \frac{1}{\sqrt{2 - \sin \theta}}$, $\sin \theta \sqrt{2 - \sin^2 \theta} = \cos^2 \theta = 1 - \sin^2 \theta$.
 $\therefore \sin \theta \sqrt{2} = 1$, $\therefore \sin \theta = \frac{1}{\sqrt{2}}$; $\therefore \theta = \frac{\pi}{4}$.

CHAPTER V.

1. $\frac{a}{c} = \sin A$; $\therefore a = 5 \times \sin 30^\circ = 5 \times \frac{1}{2} = 2.5$.
2. $\frac{a}{b} = \cot B$; $\therefore a = 10 \sqrt{3} \cot 60^\circ = 10 \sqrt{3} \times \frac{1}{\sqrt{3}} = 10$.
3. $\frac{c}{b} = \sec A$; $\therefore c = 20 \sec 45^\circ = 20 \sqrt{2}$.
4. $\cos B = \frac{a}{c} = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$; $\therefore B = 30^\circ$.
5. $\tan A = \frac{a}{b} = \frac{6}{2\sqrt{3}} = \sqrt{3}$; $\therefore A = 60^\circ$.
6. $\sin B = \frac{b}{c} = \frac{5\sqrt{2}}{10} = \frac{1}{\sqrt{2}}$; $\therefore B = 45^\circ$.
7. $B = 90^\circ - A = 90^\circ - 30^\circ = 60^\circ$. $\frac{a}{c} = \sin A$; $\therefore a = 50 \sin 30^\circ =$
 $50 \times \frac{1}{2} = 25$. $\frac{b}{c} = \cos A$; $\therefore b = 50 \cos 30^\circ = 50 \times \frac{\sqrt{3}}{2} = 25 \sqrt{3}$

$$8. \angle A = 90^\circ - B = 90^\circ - 30^\circ = 60^\circ. \quad \frac{b}{a} = \tan B; \quad \therefore b = 1732 \times \frac{1}{\sqrt{3}} = \frac{1732}{1.732} = 1000. \quad \frac{c}{a} = \sec B; \quad \therefore c = 1732 \times \frac{2}{\sqrt{3}} = \frac{1732 \times 2}{1.732} = 2000.$$

$$9. \tan B = \frac{b}{a} = \frac{43.3}{25} = 1.732 = \sqrt{3}; \quad \therefore B = 60^\circ.$$

$$\angle A = 90^\circ - B = 30^\circ. \quad \frac{c}{a} = \sec B; \quad \therefore c = 25 \times 2 = 50.$$

$$10. \sec A = \frac{c}{b} = \frac{50.904}{36} = 1.414 = \sqrt{2}; \quad \therefore A = 45^\circ.$$

$$B = 90^\circ - A = 45^\circ. \quad \frac{a}{b} = \tan A; \quad \therefore a = 36 \times 1 = 36.$$

$$11. \frac{a}{c} = \sin A; \quad \therefore a = 100 \times .145 = 14.5.$$

$$12. \frac{a}{b} = \cot B; \quad \therefore a = 3.2 \times .256 = .8192.$$

$$13. \cos A = \frac{b}{c} = \frac{15.5}{25} = .62; \quad \therefore A = 51^\circ 41'.$$

$$14. \tan B = \frac{b}{a} = \frac{5529\frac{1}{2}}{8333\frac{1}{3}} = \frac{16589}{25000} = .66356; \quad \therefore B = 33^\circ 34'.$$

$$15. \text{Taking fig. (6), } CA = CD \sec ACD = 433 \times 2 = 866. \quad CB = CD \sec BCD = 433 \frac{2}{\sqrt{3}} = \frac{866}{1.732} = 500. \quad AB = AD + DB = CD \tan ACD + CD \tan BCD = 433 \left(\sqrt{3} + \frac{1}{\sqrt{3}} \right) = 433 \frac{4}{\sqrt{3}} = \frac{1732}{1.732} = 1000.$$

$$16. \text{Taking fig. (15), } PQH = 45^\circ = PHQ; \quad \therefore PQ = PH = x \text{ suppose; then } PR = x + 73.2; \text{ and } \therefore \frac{PR}{PH} = \cot 30^\circ, \therefore \frac{x + 73.2}{x} = \sqrt{3} = 1.732; \quad \therefore x(1.732 - 1) = 73.2, \text{ or } .732 \times x = 73.2; \quad \therefore x = 100.$$

$$17. \text{In fig. to Art. 30, (2), } CD = BC \cot \beta; \quad \therefore AC = CD \tan \alpha = BC \tan \alpha \cot \beta = b \frac{\tan \alpha}{\tan \beta}.$$

$$18. \text{Let } CD = x; \text{ then } BC = x \tan \beta; \quad \therefore AC = x \tan \beta + a; \text{ and } AC = x \tan \alpha; \quad \therefore x \tan \alpha = x \tan \beta + a; \quad \therefore x(\tan \alpha - \tan \beta) = a; \quad \therefore x = \frac{a}{\tan \alpha - \tan \beta}.$$

$$19. DBC = 90^\circ - BDC = 45^\circ = BDC; \quad \therefore CD = BD \cos 45^\circ = \frac{70.7}{1.414} = 50; \quad \therefore \tan ADC = \frac{CA}{CD} = \frac{86.6}{50} = 1.732 = \sqrt{3}; \quad \therefore ADC = 60^\circ; \quad \therefore BDA = 60^\circ - 45^\circ = 15^\circ.$$

20. In $\triangle ABC$, fig. (14), each angle $= 60^\circ$; and $\therefore BC = BD$, $RCD = D$.
 $\therefore 2D = BCD + D = ABC = 60^\circ$; $\therefore ACD = 60^\circ + 30^\circ = 90^\circ$;
 $\therefore CD = AC \tan A = a \tan 60^\circ = a\sqrt{3}$.

CHAPTER VI.

. The references in brackets—thus, (1), (2), &c.—are to the figures of Plate I.

- In (1), let A be foot of person, CB the pillar. We have given $BC = 45$ ft., $A = 60^\circ$; required AC . $AC = CB \cot A = 45 \text{ ft.} \times \cot 60^\circ = 45 \text{ ft.} \times \frac{1}{\sqrt{3}} = \frac{15 \times 3}{\sqrt{3}} \text{ ft.} = 15\sqrt{3} \text{ ft.}$
- In (2), let BC be tower, and CA the horizontal base. Given $AC = 100$ yds., $A = 30^\circ$; required BC . $BC = AC \tan A = 100 \text{ yds.} \times \tan 30^\circ = \frac{100}{\sqrt{3}} \text{ yds.} = \frac{100\sqrt{3}}{3} \text{ yds.}$
- In (3), let S, S' be positions of ship, and L that of lighthouse, and let Se be eastward, $S'e$ southward. Given $SS' = 12$ m., $S'Se = 45^\circ$, $eSL = 45^\circ$, $eS'L = 15^\circ$; required SL and $S'L$. Evidently $S'SL = 45^\circ + 45^\circ = 90^\circ$, and the triangle is right-angled; also $S'SL = 45^\circ + 15^\circ = 60^\circ$; $\therefore SL = SS' \tan S'SL = 12 \text{ m.} \times \tan 60^\circ = 12 \text{ m.} \times \sqrt{3} = 12\sqrt{3} \text{ m.}$; and $S'L = SS' \sec S'SL = 12 \text{ m.} \times 2 = 24 \text{ m.}$
- In (4), let CB be tower, BD the steeple, and A the point of observation. Given $DAC = 60^\circ$, $BAC = 45^\circ$; reqd. $DC : BC$. $BC = AC \tan 45^\circ = AC$; $DC = AC \tan 60^\circ = AC \times \sqrt{3}$; $\therefore DC : BC = AC\sqrt{3} : AC = \sqrt{3} : 1$.
- In (5), let BD be pole, BC height of mound, A point of observation. Given $BAC = 30^\circ$, $DAC = 60^\circ$; reqd. to show $BD = 2BC$. $DC = AC \tan DAC = AC \tan 60^\circ = AC\sqrt{3}$, $BC = AC \tan 30^\circ = \frac{AC}{\sqrt{3}}$;
 $\therefore BD = DC - BC = AC \left(\sqrt{3} - \frac{1}{\sqrt{3}} \right) = AC \frac{3-1}{\sqrt{3}} = \frac{2AC}{\sqrt{3}} = 2BC$.
- In (6), let AB be canal, C the crossing of the roads, CD the path. Given $A = 30^\circ$, $B = 60^\circ$, $AC = 3$ m., CD perp. to AB ; reqd. BC, CD . $A + B = 90^\circ$, $\therefore ACB = 90^\circ$, $\therefore BC = CA \tan A = 3 \text{ m.} \times \frac{1}{\sqrt{3}} = \sqrt{3} \text{ m.}$, and $CD = CA \sin A = 3 \text{ m.} \times \frac{1}{2} = 1\frac{1}{2} \text{ m.}$
- In (7), let HC, CK be two positions of ladder, AB width of street, H, K the windows. Given $HC = CK = 30$ ft., $HA = 24$ ft., $HCK = 90^\circ$; reqd. AB, BK . $AC^2 = HC^2 - HA^2 = 900 - 576 = 324$, $\therefore AC = 18$. Also $\therefore HCK = 90^\circ$, $\therefore HCA + KCB = 90^\circ$, $\therefore KCB = 90^\circ - HCA = H$; $\therefore BC = CK \cos BCK = CH \cos H = HA = 24$ ft.; and $AB = 18 \text{ ft.} + 24 \text{ ft.} = 42 \text{ ft.}$; also $BK = CK \sin BCK = CH \sin H = AC = 18 \text{ ft.}$

8. In (8), let BC be steeple, A, D the two points of observation, DE parallel to AC . Given $AD = 30$ ft., $BAC = 45^\circ$, $BDE = 30^\circ$; reqd. BC, CA . $\therefore BAC = 45^\circ$, $\therefore AC = CB = x$ ft. suppose; then $BE = BC - AD = x - 30$; $\therefore \frac{x-30}{x} = \tan BDE = \tan 30^\circ = \frac{1}{\sqrt{3}}$;
 $\therefore x(\sqrt{3}-1) = 30\sqrt{3}$; $\therefore x = \frac{30\sqrt{3}}{\sqrt{3}-1} = \frac{30\sqrt{3}(\sqrt{3}+1)}{3-1} = 15(3+\sqrt{3})$; $\therefore AC = BC = 15(3+\sqrt{3})$ ft.
9. In (3), let $S'S, LS$ be courses of the two ships, S_e eastward, S'_e southward. Given $S'S_e = 45^\circ$, $eSL = 45^\circ$, $eS'L = 15^\circ$; reqd. $LS : S'S$. Evidently $S'SL = 45^\circ + 45^\circ = 90^\circ$, and the triangle is right-angled. Also $SS'L = 45^\circ + 15^\circ = 60^\circ$. $\therefore \frac{S'S}{LS} = \cot SS'L = \frac{1}{\sqrt{3}}$, or
 $S'S : LS = 1 : \sqrt{3}$.
10. In (10), let BC be mast, A point observed, BD horizontal. Given $BC = 125$ ft., $DBA = 6^\circ 50' 34''$; reqd. AC . $BAC = \text{alternate } \angle DBA$; $\therefore AC = CB \cot BAC = \frac{CB}{\tan BAC} = \frac{BC}{\tan DBA} = \frac{125}{.12}$ ft. = $1041\frac{2}{3}$ ft.
11. In (2), let AB be string, BC height of kite. Given $AB = 200$ yds., $A = 36^\circ$; reqd. BC . $BC = AB \sin A = AB \sin 36^\circ = 200 \text{ yds.} \times .5878 = 117.56$ yds.
12. In (11), let BC be wall, CA ditch, and AD the distance backward. Given $BC = 20$ ft., $BAC = 60^\circ$, $BDC = 30^\circ$; reqd. CA, AD . $CA = CB \cot BAC = 20 \text{ ft.} \times \frac{1}{\sqrt{3}} = 11.547$ ft. nearly. $CD = CB \cot D = 20\sqrt{3}$ ft. = 34.64 ft.; $\therefore AD = 34.64 \text{ ft.} - 11.547 \text{ ft.} = 23.093$ ft.
13. In (12), let B be steeple, C the tower, and AD the distance walked. Given $AD = 5$ m., $ADB = 45^\circ$, $ADC = 15^\circ$; reqd. BC . $AC = AD \tan ADC = AD \tan 15^\circ = AD \cot 75^\circ = 5 \text{ m.} \times .26 = 1.3 \text{ m.}$; $AB = AD = 5 \text{ m.}$; $\therefore BC = 1.3 \text{ m.} + 5 \text{ m.} = 6.3 \text{ m.}$
14. In (9), let A, B be the milestones, CD the height of the hill. Given $AB = 1760$ yds., $EDB = 12^\circ 13'$, $EDA = 2^\circ 45'$; reqd. $CD = x$ yards suppose. $AB = AC - BC = x \cot A - x \cot CBD = x(\cot ADE - \cot BDE) = x \left(\frac{1}{.048} - \frac{1}{.217} \right) = x \frac{.217 - .048}{.048 \times .217} = \frac{.169}{.010416} x = \frac{169}{10.416} x$; $\therefore x = \frac{10.416}{169} AB = \frac{10.416}{169} \times 1760 = 108.47$ nearly; $\therefore CD = 108.47$ yds.
15. $a = a$ and $a \cot A = b$; \therefore adding, $a(1 + \cot A) = a + b$; $\therefore a = \frac{a+b}{1 + \cot A}$. $e \sin A = a$, $e \cos A = b$; $\therefore e(\sin A + \cos A) = a + b$;
 $\therefore e = \frac{a+b}{\sin A + \cos A}$.

16. $a = a$, $a \tan B = b$; $\therefore a(1 - \tan B) = a - b$; $\therefore a = \frac{a-b}{1-\tan B}$.
 $c \cos B = a$, $c \sin B = b$; $\therefore c(\cos B - \sin B) = a - b$; $\therefore c = \frac{a-b}{\cos B - \sin B}$.
17. $a = a$, $a \cot A = b$, $a \operatorname{cosec} A = c$; $\therefore a(1 + \cot A + \operatorname{cosec} A) = a + b + c$;
 $\therefore a = \frac{a+b+c}{1 + \cot A + \operatorname{cosec} A}$. $c \sin A = a$, $c \cos A = b$, $c = c$;
 $\therefore c(\sin A + \cos A + 1) = a + b + c$; $\therefore c = \frac{a+b+c}{\sin A + \cos A + 1}$.
18. $a \cot A = b$, $a \operatorname{cosec} A = c$; $\therefore a(\cot A - \operatorname{cosec} A) = b - c$; $\therefore a = \frac{b-c}{\cot A - \operatorname{cosec} A}$.

CHAPTER VII.

1. (1) $360^\circ + 30^\circ = 390^\circ$. (2) $2.360^\circ + 70^\circ = 790^\circ$. (3) $3.360^\circ + 125^\circ = 1080^\circ + 125^\circ = 1205^\circ$. (4) $4.360^\circ + 300^\circ = 1440^\circ + 300^\circ = 1740^\circ$.
2. (1) At OP_3 the angle $= -(360^\circ - 125^\circ) = -235^\circ$; at OP it $= -(360^\circ - 30^\circ) = -330^\circ$. (2) $-360^\circ - 60^\circ = -420^\circ$.
 (3) $-2.360^\circ - 290^\circ = -720^\circ - 290^\circ = -1010^\circ$.
3. A line turning from OA positively would have described on coming to OB for the 1st time 90° , the 2nd time $360^\circ + 90^\circ$, the 3rd time $2.360^\circ + 90^\circ$. The line turning negatively, the angles would be -270° , $-360^\circ - 270^\circ$, and $-2.360^\circ - 270^\circ$. Hence the required angles are 90° , 450° , 810° , -270° , -630° , -990° .
4. 50° in 1st; $250^\circ = 180^\circ + 70^\circ$, and is \therefore in 3rd; $-450^\circ = -360^\circ - 90^\circ$, and is \therefore between the 4th and 3rd; $650^\circ = 360^\circ + 270^\circ + 20^\circ$, and is \therefore in 4th; $-850^\circ = -2.360^\circ - 90^\circ - 40^\circ$, and is \therefore in 3rd; $-1000^\circ = -2.360^\circ - 270^\circ - 10^\circ$, and is \therefore in 1st; $\frac{4}{3}\pi = \frac{2}{3}\pi + \frac{1}{3}\pi$, and is \therefore in 4th; $-\frac{4}{3}\pi = -\pi - \frac{1}{3}\pi$, and is \therefore in 2nd.
5. Let $\angle OP_2 = A$; then $\sin A = \frac{P_2 N_2}{OP_2} = \frac{4}{5}$; $\cos A = \frac{ON_2}{OP_2} = \frac{-3}{5} = -\frac{3}{5}$.
 $\tan A = \frac{4}{-3} = -\frac{4}{3}$; $\cot A = \frac{-3}{4} = -\frac{3}{4}$; $\sec A = \frac{5}{-3} = -\frac{5}{3}$;
 $\operatorname{cosec} A = \frac{5}{4}$; $\operatorname{vers} A = 1 + \frac{3}{5} = \frac{8}{5}$.
6. $\sin A = \frac{-y}{r} = -\frac{y}{r}$; $\cos A = \frac{-x}{r} = -\frac{x}{r}$; $\tan A = \frac{-y}{-x} = \frac{y}{x}$;
 $\cot A = \frac{-x}{-y} = \frac{x}{y}$; $\sec A = \frac{r}{-x} = -\frac{r}{x}$; $\operatorname{cosec} A = \frac{r}{-y} = -\frac{r}{y}$;
 $\operatorname{vers} A = 1 + \frac{x}{r} = \frac{r+x}{r}$.

7. Let A be any angle (fig. to Art. 38); then, whatever position PN may have, $\sin A = \frac{PN}{OP}$ and $\operatorname{cosec} A = \frac{OP}{PN}$, by definition of these functions; $\therefore \sin A = \frac{1}{\operatorname{cosec} A}$ always. For all positions of ON , by defs., $\cos A = \frac{ON}{OP}$ and $\sec A = \frac{OP}{ON}$; \therefore always $\cos A = \frac{1}{\sec A}$.
8. In all positions of OP , PON is a right-angled triangle having OP the hypotenuse; \therefore by Geometry, always $PN^2 + ON^2 = OP^2$; \therefore
 $\left(\frac{PN}{OP}\right)^2 + \left(\frac{ON}{OP}\right)^2 = 1$, or $\sin^2 A + \cos^2 A = 1$. Also $\left(\frac{PN}{ON}\right)^2 + 1 = \left(\frac{OP}{ON}\right)^2$ or $\tan^2 A + 1 = \sec^2 A$.

CHAPTER VIII.

1. Construct as in Art. 45. Then $\cot A = \frac{x}{y}$; hence

When A is	0° or 360°	in 1st Quadrant	90°	in 2nd Quadrant	180°	in 3rd Quadrant	270°	in 4th Quadrant
x is	r	+ve and decreasing	0	-ve and increasing	$-r$	-ve and decreasing	0	+ve and increasing
y is	0	+ve and increasing	r	+ve and decreasing	0	-ve and increasing	$-r$	-ve and decreasing
$\therefore \cot A$ is	∞	+ve and decreasing	0	-ve and increasing	∞	+ve and decreasing	0	-ve and increasing

2. Evidently, in fig. to Art. 45, when $\angle AOP$ is $< 45^\circ$, we have, in the $\triangle NOP$, $\angle OPN > \angle PON$, and \therefore side $ON > PN$, $\therefore \frac{ON}{OP} > \frac{PN}{OP}$, or $\cos A > \sin A$. When $\angle AOP$ is between 45° and 135° , we have, in the $\triangle NOP$, $\angle PON > \angle OPN$, and \therefore side $PN > ON$, and $\therefore \frac{PN}{OP} > \frac{ON}{OP}$, or $\sin A > \cos A$.
3. In same fig., between 135° and 225° , we have, in $\triangle PON$, the $\angle PON < 45^\circ$, and $\therefore \angle OPN > 45^\circ$; $\therefore \angle PON$ is $< \angle OPN$, and $ON > PN$, and $\therefore \frac{ON}{PN} > \frac{PN}{ON}$, or $\cot A > \tan A$.
4. In same fig., $\sin A = \frac{PN}{OP}$, and in the right-angled triangle PON , OP is always the greatest side, being opposite to the right angle, $\therefore \frac{PN}{OP}$ cannot be > 1 . Again, $\sec A = \frac{OP}{ON}$, which is never less than 1 for the reason just given.

5. In same fig., $\tan A = \frac{PN}{ON}$; and when A is near 90° , PN is very great and ON very small; but the latter changes sign on A passing through 90° ; hence $\tan A_1$ and $\tan A_2$ are both very great, but the former +ve and the latter -ve. $\sec A = \frac{OP}{ON}$, of which OP has a fixed magnitude and ON is as just described; $\therefore \sec A_1$ and $\sec A_2$ are both very great but differ in sign.
6. $\cot A$ is very great in magnitude when ON is very great and PN very small and when they differ in sign. This happens when A approaches 180° and 360° closely.
7. $\operatorname{cosec} A$ is very large and -ve when PN is very small and -ve, and \therefore when A has just passed through 180° and also when it is approaching closely to 360° .
8. Construct as in Art. 45. $\sec A = \frac{r}{x}$, in which x alone varies. x varies from $-r$ at OA' through 0 to $+r$ at OA ; after which it decreases again through 0 to $-r$. Hence the range of $\frac{r}{x}$ is from -1 through ∞ to $+1$, and the opposite.
9. $\operatorname{cosec} A = \frac{r}{y}$, in which y alone varies; y is greatest when it $= \pm r$, $\therefore \operatorname{cosec} A$ is least when it $= \pm 1$.
10. In fig. to Art. 45, $\tan A = \frac{y}{x}$, in which y and x both vary, but oppositely, the one increasing as the other decreases. As y is finite when x is approaching 0, and made as small as we like, $\frac{y}{x}$ is then as large as we like. As x is finite when y is as small as we like, $\frac{y}{x}$ may be taken as small as we like.
11. Vers $A = 1 - \cos A$; and assuming the variations of $\cos A$ as known, and that $\cos A$ is never > 1 ,

When A is	0 or 2π	in 1st Quadrant	$\frac{1}{2}\pi$	in 2nd Quadrant	π	in 3rd Quadrant	$\frac{3}{2}\pi$	in 4th Quadrant
$\cos A$ is	1	+ve and decreasing	0	-ve and increasing	-1	-ve and decreasing	0	+ve and increasing
\therefore vers A is	0	+ve and increasing	1	+ve and increasing	2	+ve and decreasing	1	+ve and decreasing

12. $\sec^2 \theta$ must vary similarly to $\sec \theta$, except that it increases and decreases more rapidly from being a square and is never negative. Hence we may say, assuming the variation of $\sec \theta$ as known,

When A is	0 or 2π	in 1st Quadrant	$\frac{1}{2}\pi$	in 2nd Quadrant	π	in 3rd Quadrant	$\frac{3}{2}\pi$	in 4th Quadrant
$\sec^2 \theta$ is	1	increasing	∞	decreasing	1	increasing	∞	decreasing

CHAPTER IX.

1. $\cos 160^\circ = -\cos (180^\circ - 160^\circ) = -\cos 20^\circ$. $\tan 260^\circ = \tan (180^\circ + 80^\circ) = \tan 80^\circ$. $\sec (-200^\circ) = \sec 200^\circ = \sec (180^\circ + 20^\circ) = -\sec 20^\circ$. $\operatorname{cosec} (-1000^\circ) = \operatorname{cosec} (3 \cdot 360^\circ - 1000^\circ) = \operatorname{cosec} 80^\circ$.
2. $\sin (\frac{3}{2}\pi - \alpha) = \sin [\pi + (\frac{1}{2}\pi - \alpha)] = -\sin (\frac{1}{2}\pi - \alpha) = -\cos \alpha$;
 $\tan (\frac{3}{2}\pi - \alpha) = \tan (\frac{1}{2}\pi - \alpha) = \cot \alpha$; $\cot (\frac{3}{2}\pi + \theta) = -\cot [2\pi - (\frac{1}{2}\pi + \theta)]$
 $= -\cot (\frac{1}{2}\pi - \theta) = -\tan \theta$; $\sec (\frac{3}{2}\pi + \theta) = \sec (\frac{1}{2}\pi - \theta) = \operatorname{cosec} \theta$.
3. $\cos 135^\circ = -\cos 45^\circ = -\frac{1}{\sqrt{2}}$; $\cot 225^\circ = \cot 45^\circ = 1$;
 $\operatorname{cosec} (-510^\circ) = -\operatorname{cosec} 510^\circ = -\operatorname{cosec} 150^\circ = -\operatorname{cosec} 30^\circ = -2$.
4. $\sin (2n\pi + \frac{3}{2}\pi) = \sin \frac{3}{2}\pi = \sin \frac{1}{2}\pi = \frac{1}{2}\sqrt{3}$; $\cos (-2n\pi + \frac{1}{2}\pi) = \cos \frac{1}{2}\pi = \frac{1}{2}\sqrt{3}$; $\tan (2n\pi + \pi - \frac{1}{2}\pi) = \tan (\pi - \frac{1}{2}\pi) = -\tan \frac{1}{2}\pi = -1$;
 $\cot [2n\pi + (\frac{3}{2}\pi - \frac{2}{3}\pi)] = \cot \frac{5}{6}\pi = -\cot \frac{1}{6}\pi = -\frac{1}{\sqrt{3}}$.
5. $-\frac{1}{2}\sqrt{3} = -\cos 30^\circ = \cos (180^\circ - 30^\circ)$, or $\cos (180^\circ + 30^\circ) = \cos 150^\circ$ or $\cos 210^\circ$, and $\cos 150^\circ = \cos (360^\circ + 150^\circ)$; \therefore required angles are $150^\circ, 210^\circ, 510^\circ$.
6. $\cot^2 \theta = 1$, $\therefore \cot \theta = \pm 1$; and $1 = \cot \frac{1}{2}\pi = \cot (\pi + \frac{1}{2}\pi)$; also $-1 = \cot (\pi - \frac{1}{2}\pi)$ or $\cot (2\pi - \frac{1}{2}\pi)$; \therefore the four least positive angles are $\frac{1}{2}\pi, \frac{3}{2}\pi, \frac{5}{2}\pi, \frac{7}{2}\pi$; the lines which contain these also contain $-\frac{1}{2}\pi, -\frac{3}{2}\pi, -\frac{5}{2}\pi, -\frac{7}{2}\pi$, which are the least negative angles.
7. $2\sin^2 A + \sin A - 1 = 0$; $\therefore \sin A = \frac{1}{2}[-1 \pm \sqrt{(1+8)}] = \frac{1}{2}(-1 \pm 3) = \frac{1}{2}$ or -1 , and $\frac{1}{2} = \sin 30^\circ = \sin 150^\circ$, $-1 = \sin 270^\circ$; \therefore required angles are $30^\circ, 150^\circ, 270^\circ$.
8. $2(1 - \cos^2 A) = 3 \cos A$, $\therefore 2 \cos^2 A + 3 \cos A - 2 = 0$, $\therefore \cos A = \frac{1}{2}[-3 \pm \sqrt{(9+16)}] = \frac{1}{2}(-3 \pm 5) = \frac{1}{2}$ or -2 ; and $\frac{1}{2} = \cos 60^\circ = \cos 300^\circ$, -2 is not a cosine; \therefore the angles are $60^\circ, 300^\circ$.
9. $\tan^2 \theta + \frac{1}{\tan^2 \theta} = \frac{10}{3}$, $\therefore 3 \tan^4 \theta - 10 \tan^2 \theta + 3 = 0$, $\therefore \tan^2 \theta = \frac{1}{3}[10 \pm \sqrt{(100-36)}] = \frac{1}{3}(10 \pm 8) = 3$ or $\frac{1}{3}$, $\therefore \tan \theta = \pm \sqrt{3}$ or $\pm \frac{1}{\sqrt{3}}$; \therefore angles are $60^\circ, 120^\circ, 30^\circ, 150^\circ$.
10. Let A be the angle. Then $\sin (180^\circ - A) = \sin (90^\circ - A)$, $\therefore \sin A = \cos A$, $\therefore A = 45^\circ$.
11. $\cos \left(\frac{0 \cdot \pi}{2} - \theta \right) = \cos (-\theta) = \cos \theta$; $\cos \left(\frac{1 \cdot \pi}{2} - \theta \right) = \sin \theta$.
 $\cos \left(\frac{2 \cdot \pi}{2} - \theta \right) = \cos (\pi - \theta) = -\cos \theta$; $\cos \left(\frac{3 \cdot \pi}{2} - \theta \right) = -\cos \left(\frac{1}{2}\pi - \theta \right) = -\sin \theta$, $\cos \left(\frac{4 \cdot \pi}{2} - \theta \right) = \cos (2\pi - \theta) = \cos (-\theta)$,
which is the first case over again. The rest come over again on taking higher values of n .
12. $\sin A = \cos (90^\circ - A) = \cos [-(90^\circ - A)] = \cos (A - 90^\circ)$,
 $\cos A = \sin (90^\circ - A) = -\sin [-(90^\circ - A)] = -\sin (A - 90^\circ)$.

EXERCISE V.

$$2. \quad l = \cos 90^\circ \cos A + \sin 90^\circ \sin A = f; \quad l = \sin 90^\circ \cos A \pm \cos 90^\circ \sin A = f;$$

$$l = -(\sin 2\pi \cos \theta - \cos 2\pi \sin \theta) = f.$$

$$\cos \frac{1}{2}\pi = \cos \frac{1}{4}\pi \cos \frac{1}{4}\pi - \sin \frac{1}{4}\pi \sin \frac{1}{4}\pi = \left(\frac{1}{\sqrt{2}}\right)^2 - \left(\frac{1}{\sqrt{2}}\right)^2 = 0.$$

$$\cos 0 = \cos \frac{1}{4}\pi \cos \frac{1}{4}\pi + \sin \frac{1}{4}\pi \sin \frac{1}{4}\pi = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = 1.$$

$$3. \quad f = \sin(45^\circ - 30^\circ) = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}; \text{ and}$$

$$\sin 15^\circ = \cos(90^\circ - 15^\circ) = \cos 75^\circ.$$

$$f = \cos(45^\circ - 30^\circ) = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}; \text{ and}$$

$$\cos 15^\circ = \sin(90^\circ - 15^\circ) = \sin 75^\circ.$$

EXERCISE VI.

Answers to 2 and 3 are written out at sight.

$$5. \quad \text{Let } \alpha = 22\frac{1}{2}^\circ, \text{ then } 2 \cos^2 \alpha = 1 + \cos 2\alpha = 1 + \frac{1}{\sqrt{2}} = 1 + \frac{\sqrt{2}}{2} = \frac{2 + \sqrt{2}}{2}, \therefore \cos \alpha = \frac{1}{2}\sqrt{2 + \sqrt{2}}.$$

$$6. \quad \text{Let } \alpha = 11\frac{1}{4}^\circ, \text{ then } 2 \sin^2 \alpha = 1 - \cos 2\alpha = 1 - \frac{1}{2}\sqrt{2 + \sqrt{2}} = \frac{1}{2}[2 - \sqrt{2 + \sqrt{2}}], \therefore \sin \alpha = \frac{1}{2}\sqrt{2 - \sqrt{2 + \sqrt{2}}}.$$

$$7. \quad \cot(A+B) = \frac{\cos A \cos B - \sin A \sin B}{\sin A \cos B + \cos A \sin B} = \frac{\cot A \cot B - 1}{\cot A + \cot B}, \text{ on dividing numerator and denominator by } \sin A \sin B.$$

$$\cot(A-B) = \frac{\cos A \cos B + \sin A \sin B}{\sin A \cos B - \cos A \sin B} = \frac{\cot A \cot B + 1}{\cot B - \cot A}, \text{ similarly.}$$

$$\cot 2A = \cot(A+A) = \frac{\cot A \cot A - 1}{\cot A + \cot A} = \frac{\cot^2 A - 1}{2 \cot A}.$$

$$8. \quad \tan 75^\circ = \tan(45^\circ + 30^\circ) = \frac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}+1}{\sqrt{3}-1} = \frac{(\sqrt{3}+1)^2}{3-1} = 2 + \sqrt{3}.$$

CHAPTER X.

$$1. \quad f = \sin 45^\circ \cos A + \cos 45^\circ \sin A = \frac{1}{\sqrt{2}} \cos A + \frac{1}{\sqrt{2}} \sin A = l.$$

$$2. \quad f = \cos 30^\circ \cos A + \sin 30^\circ \sin A = \frac{1}{2}\sqrt{3} \cos A + \frac{1}{2} \sin A = l.$$

$$3. \quad f = 1 - \cos(A+30^\circ) - [1 - \cos(A-30^\circ)] = \cos(A-30^\circ) - \cos(A+30^\circ) = \cos A \cos 30^\circ + \sin A \sin 30^\circ - \cos A \cos 30^\circ + \sin A \sin 30^\circ = 2 \times \frac{1}{2} \sin A = l.$$

$$4. f = \frac{2 \sin^2 A}{2 \sin A \cos A} = \frac{\sin A}{\cos A} = i.$$

$$5. f = \frac{2 \sin^2 A}{2 \cos^2 A} = i.$$

$$6. f = \frac{\tan 45^\circ + \tan A}{1 - \tan 45^\circ \tan A} = i.$$

$$7. f = \frac{\tan A - \tan 45^\circ}{1 + \tan A \tan 45^\circ} = i.$$

$$8. f = \frac{\sin A}{\cos A} + \frac{\cos A}{\sin A} = \frac{\sin^2 A + \cos^2 A}{\cos A \sin A} = \frac{2}{2 \sin A \cos A} = \frac{2}{\sin 2A} = i.$$

$$9. f = \frac{\sin A}{\cos A} - \frac{\cos A}{\sin A} = \frac{\sin^2 A - \cos^2 A}{\cos A \sin A} = \frac{-2(\cos^2 A - \sin^2 A)}{2 \sin A \cos A} = \frac{-2 \cos 2A}{\sin 2A} = i.$$

$$10. f = \frac{\sin A}{\cos A} - \frac{\sin B}{\cos B} = \frac{\sin A \cos B - \cos A \sin B}{\cos A \cos B} = i.$$

$$11. f = \frac{\cos A}{\sin A} - \frac{\sin B}{\cos B} = \frac{\cos A \cos B - \sin A \sin B}{\sin A \cos B} = i.$$

$$12. i = \frac{\cos A}{1 + \frac{\sin^2 A}{\cos^2 A}} = \frac{\cos A}{\frac{\cos^2 A + \sin^2 A}{\cos^2 A}} = \frac{2 \sin A \cos A}{1} = f.$$

$$13. i = \frac{1 - \frac{\cos^2 A}{\sin^2 A}}{1 + \frac{\cos^2 A}{\sin^2 A}} = \frac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A} = f.$$

$$14. i = 1 - \frac{2}{\operatorname{cosec}^2 A} = 1 - 2 \sin^2 A = f.$$

$$15. i = \frac{\frac{\cos \frac{1}{2} A}{\sin \frac{1}{2} A} - \frac{\sin \frac{1}{2} A}{\cos \frac{1}{2} A}}{\frac{\sin \frac{1}{2} A}{\cos \frac{1}{2} A} + \frac{\cos \frac{1}{2} A}{\sin \frac{1}{2} A}} = \frac{\cos^2 \frac{1}{2} A - \sin^2 \frac{1}{2} A}{\cos^2 \frac{1}{2} A + \sin^2 \frac{1}{2} A} = f.$$

$$16. i = \cos 2(45^\circ - \frac{1}{2} A) = \cos(90^\circ - A) = f.$$

$$17. 45^\circ + A \text{ is complement of } 45^\circ - A, \\ \therefore i = 2 \sin(45^\circ - A) \cos(45^\circ - A) = \sin(90^\circ - 2A) = f.$$

$$18. f = 2 \left(1 - \frac{\sin A \cos 2A}{\cos A \sin 2A} \right) = 2 \frac{\cos A \sin 2A - \sin A \cos 2A}{\cos A \sin 2A} \\ = \frac{2 \sin(2A - A)}{\cos A \cdot 2 \sin A \cos A} = \frac{1}{\cos^2 A} = i.$$

$$19. i = \frac{1}{1 + \frac{\sin 2A \sin 4A}{\cos 2A \cos 4A}} = \frac{\cos 2A \cos 4A}{\cos 2A \cos 4A + \sin 2A \sin 4A} \\ = \frac{\cos 2A \cos 4A}{\cos(4A - 2A)} = f.$$

$$20. f = (\sin A \cos B + \cos A \sin B)(\sin A \cos B - \cos A \sin B) = \sin^2 A \cos^2 B \\ - \cos^2 A \sin^2 B = \sin^2 A (1 - \sin^2 B) - (1 - \sin^2 A) \sin^2 B = i.$$

21. $f = \sin^2 A \cos^2 B - \cos^2 A \sin^2 B$
 $= (1 - \cos^2 A) \cos^2 B - \cos^2 A (1 - \cos^2 B) = l.$
22. $f = \cos^2 A \cos^2 B - \sin^2 A \sin^2 B$
 $= \cos^2 A (1 - \sin^2 B) - (1 - \cos^2 A) \sin^2 B = l.$
23. $f = \cos^2 A \cos^2 B - \sin^2 A \sin^2 B$
 $= (1 - \sin^2 A) \cos^2 B - \sin^2 A (1 - \cos^2 B) = l.$
24. $f = \frac{\cos \theta \cos \frac{1}{2}\pi - \sin \theta \sin \frac{1}{2}\pi}{\cos \theta \cos \frac{1}{2}\pi + \sin \theta \sin \frac{1}{2}\pi} = \frac{\frac{1}{\sqrt{2}}(\cos \theta - \sin \theta)}{\frac{1}{\sqrt{2}}(\cos \theta + \sin \theta)} = \frac{(\cos \theta - \sin \theta)^2}{\cos^2 \theta - \sin^2 \theta}$
 $= \frac{\cos^2 \theta + \sin^2 \theta - 2 \sin \theta \cos \theta}{\cos 2\theta} = \frac{1 - \sin 2\theta}{\cos 2\theta} = l.$
25. $l = \left\{ \frac{\tan \frac{1}{2}\pi + \tan \frac{1}{2}\alpha}{1 - \tan \frac{1}{2}\pi \tan \frac{1}{2}\alpha} \right\}^2 = \left\{ \frac{1 + \frac{\sin \frac{1}{2}\alpha}{\cos \frac{1}{2}\alpha}}{1 - \frac{\sin \frac{1}{2}\alpha}{\cos \frac{1}{2}\alpha}} \right\}^2 = \left\{ \frac{\cos \frac{1}{2}\alpha + \sin \frac{1}{2}\alpha}{\cos \frac{1}{2}\alpha - \sin \frac{1}{2}\alpha} \right\}^2$
 $= \frac{\cos^2 \frac{1}{2}\alpha + \sin^2 \frac{1}{2}\alpha + 2 \sin \frac{1}{2}\alpha \cos \frac{1}{2}\alpha}{\cos^2 \frac{1}{2}\alpha + \sin^2 \frac{1}{2}\alpha - 2 \sin \frac{1}{2}\alpha \cos \frac{1}{2}\alpha} = f.$
26. $l = \sin \theta \frac{1}{\sec 2\phi} + \cos \theta \frac{1}{\operatorname{cosec} 2\phi} = \sin \theta \cos 2\phi + \cos \theta \sin 2\phi = f.$
27. $f = \frac{\sin(\alpha + 30^\circ) \sin(\alpha - 30^\circ)}{\cos(\alpha + 30^\circ) \cos(\alpha - 30^\circ)} = \frac{\sin^2 \alpha - \sin^2 30^\circ}{\cos^2 \alpha - \sin^2 30^\circ} = \frac{\sin^2 \alpha - \frac{1}{4}}{\cos^2 \alpha - \frac{1}{4}}$
 $= \frac{4 \sin^2 \alpha - 1}{4 \cos^2 \alpha - 1} = \frac{1 - 2(1 - 2 \sin^2 \alpha)}{1 + 2(2 \cos^2 \alpha - 1)} = l.$
28. $f = (\cos^2 \alpha - \sin^2 \alpha)(\cos^4 \alpha + \cos^2 \alpha \sin^2 \alpha + \sin^4 \alpha) =$
 $\cos 2\alpha [(\cos^2 \alpha + \sin^2 \alpha)^2 - \cos^2 \alpha \sin^2 \alpha] = \cos 2\alpha \left(1 - \frac{4 \sin^2 \alpha \cos^2 \alpha}{4} \right) = l.$
29. $l = \sin(\alpha + \gamma) \cos 2(\beta + \gamma) \frac{\cos(\alpha + \gamma)}{\sin(\alpha + \gamma)} - \cos(\alpha + \gamma) \sin 2(\beta + \gamma) \times$
 $\frac{\sin(\alpha + \gamma)}{\cos(\alpha + \gamma)} = \cos 2(\beta + \gamma) \cos(\alpha + \gamma) - \sin 2(\beta + \gamma) \sin(\alpha + \gamma)$
 $= \cos(2\beta + 2\gamma + \alpha + \gamma) = f.$
30. $f = \frac{\sin(A+B) \cos C + \cos(A+B) \sin C}{\cos A \cos B \cos C} =$
 $\frac{\sin A \cos B \cos C + \cos A \sin B \cos C + \cos A \cos B \sin C - \sin A \sin B \sin C}{\cos A \cos B \cos C} = l.$
31. $f = \frac{\cos(A+B) \cos C - \sin(A+B) \sin C}{\cos A \cos B \cos C} =$
 $\frac{\cos A \cos B \cos C - \sin A \sin B \cos C - \sin A \cos B \sin C - \cos A \sin B \sin C}{\cos A \cos B \cos C} = l.$
32. $f =$
 $\frac{\sin A \cos B \cos C + \cos A \sin B \cos C + \cos A \cos B \sin C - \sin A \sin B \sin C}{\sin A \sin B \sin C} = l.$

33. $f =$

$$\frac{\cos A \cos B \cos C - \sin A \sin B \cos C - \sin A \cos B \sin C - \cos A \sin B \sin C}{\sin A \sin B \sin C} = 1.$$

34. $f =$

$$\frac{\sin A \cos B \cos C + \cos A \sin B \cos C + \cos A \cos B \sin C - \sin A \sin B \sin C}{\cos A \cos B \cos C - \sin A \sin B \cos C - \sin A \cos B \sin C - \cos A \sin B \sin C}$$

$$= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan A \tan C - \tan B \tan C}$$

(by dividing numerator and denominator by $\cos A \cos B \cos C$).35. $f =$

$$\frac{\cos A \cos B \cos C - \sin A \sin B \cos C - \sin A \cos B \sin C - \cos A \sin B \sin C}{\sin A \cos B \cos C + \cos A \sin B \cos C + \cos A \cos B \sin C - \sin A \sin B \sin C}$$

$$= \frac{\cot A \cot B \cot C - \cot C - \cot B - \cot A}{\cot B \cot C + \cot A \cot C + \cot A \cot B - 1}$$

(by dividing numerator and denominator by $\sin A \sin B \sin C$).

$$36. f = \tan(2A + A) = \frac{\tan 2A + \tan A}{1 - \tan 2A \tan A} = \frac{\frac{2 \tan A}{1 - \tan^2 A} + \tan A}{1 - \frac{2 \tan A \tan A}{1 - \tan^2 A}} = 1.$$

Let $A = 30^\circ$, then $\tan 3A = \infty$, and $1 - 3 \tan^2 A = 0$, $\therefore \tan A = \frac{1}{\sqrt{3}}$.Let $A = 60^\circ$, then $\tan 3A = 0$, and $\tan A (3 - \tan^2 A) = 0$, $\therefore 3 - \tan^2 A = 0$, and $\tan A = \sqrt{3}$.

$$37. f = \frac{\cos A \cos B - \sin A \sin B}{\sin A \cos B + \cos A \sin B} = \frac{1 - \tan A \tan B}{\tan A + \tan B}$$

(by dividing numerator and denominator by $\cos A \cos B$).

$$\text{Hence } \cot 75^\circ = \cot(45^\circ + 30^\circ) = \frac{1 - \tan 45^\circ \tan 30^\circ}{\tan 45^\circ + \tan 30^\circ} = \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}}$$

$$= \frac{\sqrt{3} - 1}{\sqrt{3} + 1} = 2 - \sqrt{3}.$$

$$38. f = \frac{1}{\cos(A+B)} = \frac{1}{\cos A \cos B - \sin A \sin B} = \frac{\sec A \sec B}{1 - \tan A \tan B}$$

$$= \frac{\sec A \sec B}{1 - \sqrt{(\sec^2 A - 1)(\sec^2 B - 1)}}.$$

$$39. \sin 2A = \frac{1}{2}, \therefore \tan A = \frac{\sin A}{\cos A} = \frac{2 \sin A \cos A}{2 \cos^2 A} = \frac{\sin 2A}{1 + \cos 2A}$$

$$= \frac{\sin 2A}{1 + \sqrt{(1 - \sin^2 2A)}} = \frac{1}{2} + [1 + \sqrt{(1 - \frac{1}{4})}] = \frac{1}{2} + (1 + \frac{1}{2}\sqrt{3})$$

$$= \frac{1}{2} \times \frac{2}{2 + \sqrt{3}} = 2 - \sqrt{3}.$$

$$40. \sin 54^\circ = \cos (90^\circ - 54^\circ) = \cos 36^\circ = 1 - 2 \sin^2 18^\circ \\ = 1 - 2 \frac{6-2\sqrt{5}}{16} = \frac{8-6+2\sqrt{5}}{8} = \frac{2(\sqrt{5}+1)}{8} = \frac{\sqrt{5}+1}{4}.$$

$$41. 1 - 2 \sin^2 36^\circ = \cos 72^\circ = \frac{\sqrt{5}-1}{4}, \therefore 2 \sin^2 36^\circ = 1 - \frac{\sqrt{5}-1}{4} = \\ \frac{5-\sqrt{5}}{4}, \therefore \sin^2 36^\circ = \frac{5-\sqrt{5}}{8}, \therefore \sin 36^\circ = \frac{\sqrt{5-\sqrt{5}}}{2\sqrt{2}}.$$

$$42. \sin (A-B) = \sin [A+(-B)] = \sin A \cos (-B) + \cos A \sin (-B) \\ = \sin A \cos B - \cos A \sin B.$$

$$43. \sin (A-B) = -\cos [(90^\circ + A) - B] \\ = -[\cos (90^\circ + A) \cos B + \sin (90^\circ + A) \sin B] = \sin A \cos B - \cos A \sin B.$$

Let $A=B$, then $1 = \cos (A-A) = \cos A \cos A + \sin A \sin A = \cos^2 A + \sin^2 A$.

$$44. A+B < 90^\circ, \therefore A < 90^\circ - B, \therefore \cot A > \cot (90^\circ - B) > \tan B \\ > \frac{1}{\cot B}, \therefore \cot A \cot B > 1.$$

EXERCISE VII.

2. (i.) $f = \sin (\theta + \phi + \theta - \phi) - \sin (\theta + \phi - \theta + \phi) = l$.
 (ii.) $f = \sin (A + 2B + A - 2B) + \sin (A + 2B - A + 2B) = l$.
 (iii.) $f = \cos (2\alpha + \beta + \alpha + 2\beta) + \cos (2\alpha + \beta - \alpha - 2\beta) = l$.
 (iv.) $f = \cos (4A - 3A) - \cos (4A + 3A) = l$.
 (v.) $f = \sin (\frac{1}{2}\pi + A + \frac{1}{2}\pi - A) + \sin (\frac{1}{2}\pi + A - \frac{1}{2}\pi + A) \\ = \sin \frac{1}{2}\pi + \sin 2A = l$.
 (vi.) $f = \cos (\frac{1}{2}\pi + \frac{1}{2}A + \frac{1}{2}\pi - \frac{1}{2}A) + \cos (\frac{1}{2}\pi + \frac{1}{2}A - \frac{1}{2}\pi + \frac{1}{2}A) \\ = \cos \pi + \cos A = l$.
 (vii.) $f = 2 [\cos (60^\circ + A - 60^\circ + A) - \cos (60^\circ + A + 60^\circ - A)] \\ = 2 (\cos 2A - \cos 120^\circ) = l$.
 (viii.) $f = \cos (\frac{1}{\sqrt{3}}\theta - \frac{1}{\sqrt{3}}\theta) - \cos (\frac{1}{\sqrt{3}}\theta + \frac{1}{\sqrt{3}}\theta) = l$.
4. (i.) $f = 2 \sin \frac{1}{2} (30^\circ + A + 30^\circ - A) \sin \frac{1}{2} (30^\circ + A - 30^\circ + A) \\ = 2 \sin 30^\circ \sin A = l$.
 (ii.) $f = 2 \sin \frac{1}{2} (60^\circ + A + 60^\circ - A) \cos \frac{1}{2} (60^\circ + A - 60^\circ + A) \\ = 2 \sin 60^\circ \cos A = l$.
 (iii.) $f = 2 \cos \frac{1}{2} (\frac{1}{2}\pi + \theta + \frac{1}{2}\pi - \theta) \sin \frac{1}{2} (\frac{1}{2}\pi + \theta - \frac{1}{2}\pi + \theta) \\ = 2 \cos \frac{1}{2}\pi \sin \theta = l$.
 (iv.) $f = \frac{2 \cos \frac{1}{2} (2\alpha - 3\beta + 5\beta) \cos \frac{1}{2} (2\alpha - 3\beta - 5\beta)}{2 \sin \frac{1}{2} (2\alpha - 3\beta + 5\beta) \cos \frac{1}{2} (2\alpha - 3\beta - 5\beta)} = \frac{\cos (\alpha + \beta)}{\sin (\alpha + \beta)} = l$.

CHAPTER XI.

1. $f = 2 \cos 60^\circ \cos A = 2 \times \frac{1}{2} \cos A = l$. 2. $l = 2 \sin \frac{1}{2}A \cos \frac{1}{2}A = \\ \sin \frac{1}{2}A + \sin \frac{3}{2}A = f$. 3. $f = 2 (\cos 60^\circ + \cos 2A) = 2 (\frac{1}{2} + \cos 2A) = l$.
 4. $f = 2 \cos 2B \sin A = l$. 5. $l = 2 \sin 3A \cos A - \sin 3A = \\ \sin 4A + \sin 2A - \sin 3A = f$. 6. $l = 2 \cos \frac{1}{2}\theta \cos \frac{1}{2}\theta + \cos \frac{3}{2}\theta = f$.

- $$7. f = \frac{2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)}{2 \cos \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)} = \frac{\sin \frac{1}{2}(A+B)}{\cos \frac{1}{2}(A+B)} = l.$$
- $$8. f = \frac{2 \cos \frac{1}{2}(A+B) \sin \frac{1}{2}(A-B)}{2 \cos \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)} = \frac{\sin \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A-B)} = l.$$
- $$9. f = \frac{2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)}{2 \cos \frac{1}{2}(A+B) \sin \frac{1}{2}(A-B)} = l.$$
- $$10. f = \frac{2 \sin \frac{1}{2}(A+B) \sin \frac{1}{2}(A-B)}{2 \cos \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)} = l. \quad 11. f = \frac{2 \sin 2A \sin A}{2 \cos 2A \sin A} = l.$$
- $$12. f = \frac{2 \sin A \cos B}{-2 \sin A \sin B} = l. \quad 13. f = \frac{2 \sin \frac{1}{2}\theta \cos \frac{1}{2}\theta}{2 \sin \frac{1}{2}\theta \sin \frac{1}{2}\theta} = l.$$
- $$14. l = \cos[(\alpha + \beta) + \alpha] \cos[(\alpha + \beta) - \alpha] = f. \quad (\text{See p. 60, Ex. 22.})$$
- $$15. f = \sin(3A + 2A) \sin(3A - 2A) = l. \quad (\text{See p. 60, Ex. 21.})$$
- $$16. f = \frac{\cos^2 \beta - \cos^2 \alpha}{\cos^2 \alpha \cos^2 \beta} = \frac{1}{\cos^2 \alpha} - \frac{1}{\cos^2 \beta} = \sec^2 \alpha - \sec^2 \beta$$
- $$= \tan^2 \alpha + 1 - (\tan^2 \beta + 1) = l.$$
- $$17. f = \frac{2 \sin 3\alpha \sin \alpha}{2 \cos 3\alpha \sin \alpha} = \frac{\sin 3\alpha}{\cos 3\alpha} = l.$$
- $$18. f = \frac{2 \sin \frac{3}{4}A \cos \frac{1}{4}A}{2 \cos \frac{3}{4}A \cos \frac{1}{4}A} = \frac{\sin \frac{3}{4}A}{\cos \frac{3}{4}A} = l.$$
- $$19. f = \frac{\sin \alpha + \sin 5\alpha + 2 \sin 3\alpha}{\sin 3\alpha + \sin 7\alpha + 2 \sin 5\alpha} = \frac{2 \sin 3\alpha \cos 2\alpha + 2 \sin 3\alpha}{2 \sin 5\alpha \cos 2\alpha + 2 \sin 5\alpha}$$
- $$= \frac{2 \sin 3\alpha (\cos 2\alpha + 1)}{2 \sin 5\alpha (\cos 2\alpha + 1)} = l.$$
- $$20. f = \frac{\sin 2\theta}{\cos 2\theta} - \frac{\sin \theta}{\cos \theta} = \frac{\sin 2\theta \cos \theta - \cos 2\theta \sin \theta}{\cos 2\theta \cos \theta} = \frac{2 \sin (\theta - \theta)}{2 \cos 2\theta \cos \theta} = l.$$
- $$21. f = \frac{\sin 2A}{\cos 2A} + \frac{\sin A}{\cos A} = \frac{\sin 2A \cos A + \cos 2A \sin A}{\cos 2A \cos A} = \frac{2 \sin (2A + A)}{2 \cos 2A \cos A} = l.$$
- $$22. \text{Transposing, we have } \sin(A+C) \cos C - \cos(A+C) \sin C = \sin(A+B) \cos B - \cos(A+B) \sin B, \text{ or } \sin[(A+B)-B] = \sin[(A+C)-C], \text{ which is evidently true.}$$
- $$23. 2f = \sin 2A - \sin 2B + \sin 2B - \sin 2C + \sin 2C - \sin 2D + \sin 2D - \sin 2A = 0.$$
- $$24. l = 2 \sin A \cdot 2 \sin B \sin C = 2 \sin A [\cos(B-C) - \cos(B+C)] = 2 \sin A \cos(B-C) - 2 \sin A \cos(B+C) = f.$$
- $$25. l = 2 \cos A [\cos(B+C) - \cos(B-C)] = 2 \cos A \cos(B+C) + 2 \cos A \cos(B-C) = f.$$
- $$26. f = 2 \cos A [\cos 240^\circ + \cos 2A] = 2 \cos A \times (-\frac{1}{2}) + 2 \cos A \cos 2A$$
- $$= -\cos A + \cos 3A + \cos A = l.$$
- $$27. l = 2 \cdot 2 \cos A \cos 3A \cdot 2 \cos^2 3A = 2 (\cos 4A + \cos 2A) (1 + \cos 6A)$$
- $$= 2 \cos 4A + 2 \cos 2A + 2 \cos 4A \cos 6A + 2 \cos 2A \cos 6A = f.$$
- $$28. f = \cos \theta + 2 \cos \frac{1}{2}\pi \cos \theta = \cos \theta + 2(-\frac{1}{2}) \cos \theta = 0.$$

29. $f = 2 \sin \theta (\cos 2\theta - \cos \frac{1}{2}\pi) = 2 \sin \theta \cos \theta - 2 \sin \theta \times (-\frac{1}{2})$
 $= \sin 2\theta - \sin \theta + \sin \theta = l.$
30. $f = 2 \cos^2 \theta (\sin 4\theta + \sin 2\theta) + 2 \sin^2 \theta (\sin 4\theta - \sin 2\theta) = 2 \sin 2\theta \times$
 $(\cos^2 \theta - \sin^2 \theta) + 2 \sin 4\theta (\cos^2 \theta + \sin^2 \theta) = 2 \sin 2\theta \cos 2\theta + 2 \sin 4\theta = l.$
31. $2f = \sin^2 \theta (\cos 2\theta - \cos 4\theta) + \cos^2 \theta (\cos 2\theta + \cos 4\theta)$
 $= \cos 2\theta (\cos^2 \theta + \sin^2 \theta) + \cos 4\theta (\cos^2 \theta - \sin^2 \theta)$
 $= \cos 2\theta + (2 \cos^2 2\theta - 1) \cos 2\theta = 2l.$
32. Remembering that $A = 180^\circ - (B + C),$
 $l = 2 \sin A \cdot 2 \sin B \sin C = 2 \sin A [\cos (B - C) - \cos (B + C)]$
 $= 2 \sin (B + C) \cos (B - C) + 2 \sin A \cos A$
 $= \sin 2B + \sin 2C + \sin 2A = f.$
33. Remembering $\frac{1}{2}A = 90^\circ - \frac{1}{2}(B + C),$
 $l = 1 + 2 \sin \frac{1}{2}A [\cos \frac{1}{2}(B - C) - \cos \frac{1}{2}(B + C)]$
 $= 1 + 2 \cos \frac{1}{2}(B + C) \cos \frac{1}{2}(B - C) - 2 \sin^2 \frac{1}{2}A = f.$
34. $l = 2 \sin \frac{1}{2}A [\sin \frac{1}{2}(B + C) - \sin \frac{1}{2}(B - C)]$
 $= 2 \sin \frac{1}{2}A \cos \frac{1}{2}A - 2 \cos \frac{1}{2}(B + C) \sin \frac{1}{2}(B - C) = f.$
35. $\therefore \pi - A = B + C, \text{ \&c.}, \therefore l = 2 \cos \frac{1}{2}(\pi - A) 2 \cos \frac{1}{2}(A + C) \times$
 $\cos \frac{1}{2}(A + B) = 2 \cos \frac{1}{2}(\pi - A) [\cos \frac{1}{2}(A + A + B + C) + \cos \frac{1}{2}(B - C)]$
 $= 2 \cos \frac{1}{2}(\pi - A) \cos \frac{1}{2}(\pi + A) + 2 \cos \frac{1}{2}(B + C) \cos \frac{1}{2}(B - C)$
 $= \cos \frac{1}{2}\pi + \cos \frac{1}{2}A + \cos \frac{1}{2}B + \cos \frac{1}{2}C = f.$
36. $\cos A \cdot 2 \cos B \cos C = \cos A [\cos (B + C) + \cos (B - C)]$
 $= -\cos^2 A - \cos (B + C) \cos (B - C) = -\cos^2 A - (\cos^2 B - \sin^2 C)$
 $= -\cos^2 A - \cos^2 B - \cos^2 C + 1; \therefore \text{transposing, } f = l.$
37. $f = \frac{\sin A + \sin B + \sin (A + B)}{2 [\sin \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B) - \sin \frac{1}{2}(A + B) \cos \frac{1}{2}(A + B)]}$
 $= \frac{2 [\sin \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B) + \sin \frac{1}{2}(A + B) \cos \frac{1}{2}(A + B)]}{\cos \frac{1}{2}(A - B) - \cos \frac{1}{2}(A + B)} = \frac{2 \sin \frac{1}{2}A \sin \frac{1}{2}B}{\cos \frac{1}{2}(A - B) + \cos \frac{1}{2}(A + B)} = \frac{2 \cos \frac{1}{2}A \cos \frac{1}{2}B}{2 \cos \frac{1}{2}A \cos \frac{1}{2}B} = \tan \frac{1}{2}A \tan \frac{1}{2}B.$
38. $\therefore A + B = 90^\circ - C, \therefore \tan (A + B) = \cot C,$
 $\therefore \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{1}{\tan C}, \therefore \tan A \tan C + \tan B \tan C = 1 - \tan A \tan B;$
transposing, $f = l.$
39. $1 = \sin (A + B + C) = \sin A \cos B \cos C + \cos A \sin B \cos C +$
 $\cos A \cos B \sin C - \sin A \sin B \sin C; \text{ dividing by } \cos A \cos B \cos C,$
 $\sec A \sec B \sec C = \tan A + \tan B + \tan C - \tan A \tan B \tan C, \therefore f = l.$
40. $f = \frac{2 \cos^2 A - \sin^2 A}{\cos A + \sin A} + \frac{2 \sin 2A}{\cos A - \sin A} = \frac{2 (\cos A - \sin A)}{1} + \frac{2 \sin 2A}{\cos A - \sin A}$
 $= \frac{2 \cos^2 A + \sin^2 A - 2 \sin A \cos A + \sin 2A}{\cos A - \sin A} = \frac{2}{\cos A - \sin A}$
 $= \frac{\sqrt{2}}{\frac{1}{\sqrt{2}} \cos A - \frac{1}{\sqrt{2}} \sin A} = l.$
41. $\therefore \beta + \gamma = \pi - \alpha, \therefore f = (1 - \cos \alpha)(1 + \cos \alpha) = \sin^2 \alpha = \sin \alpha \sin (\beta + \gamma)$

42. $\therefore B + C = 180^\circ - A$, $\therefore \tan A = -\tan(B + C) = -\frac{\tan B + \tan C}{1 - \tan B \tan C}$
 $\therefore \tan A - \tan A \tan B \tan C = -\tan B - \tan C$, $\therefore f = l$. Suppose the Δ equilateral, $\therefore A = B = C = 60^\circ$, $\therefore 3 \tan A = \tan^3 A$, $\therefore \tan^2 A = 3$, and $\tan A = \sqrt{3}$.
43. $\therefore \frac{a}{b} = \frac{\cos \alpha}{\cos \beta}$, $\therefore \frac{a+b}{a-b} = \frac{\cos \alpha + \cos \beta}{\cos \alpha - \cos \beta} = \frac{2 \cos \frac{1}{2}(\alpha + \beta) \cos \frac{1}{2}(\beta - \alpha)}{2 \sin \frac{1}{2}(\alpha + \beta) \sin \frac{1}{2}(\beta - \alpha)} = l$.
44. $a \cos 2\theta + b \sin 2\theta = a(1 - 2 \sin^2 \theta) + b \cdot 2 \sin \theta \cos \theta$
 $= a - 2(a \sin^2 \theta - b \sin \theta \cos \theta) = a - 2 \cos^2 \theta (a \tan^2 \theta - b \tan \theta)$
 $= a - 2 \cos^2 \theta \left(a \frac{b^2}{a^2} - b \frac{b}{a} \right) = a - 2 \cos^2 \theta \times 0 = a$.
45. (1) $\sin A + \sin B = 2 \sin \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B)$; in which $\sin \frac{1}{2}(A + B)$ is constant, and $\cos \frac{1}{2}(A - B)$ is greatest when $\frac{1}{2}(A - B) = 0$, i.e., when $A = B$. (2) $\tan A + \tan B = \frac{\sin(A + B)}{\cos A \cos B}$ [p. 58, (1)]
 $= \frac{2 \sin(A + B)}{\cos(A + B) + \cos(A - B)}$. Now numerator is constant, and in denominator $\cos(A - B)$ alone varies, and is greatest (and \therefore the fraction is least) when $A - B = 0$; i.e., when $A = B$.

EXERCISE VIII.

2. $f = b \sin A - b \sin A = l$. 3. $f = a \sin B + a \sin C = l$.
4. $f = a \sin B - b \sin A + b \sin C - c \sin B + c \sin A - a \sin C = 0$, by (2).
5. $f = b \sin A - a \sin B + c \sin B - b \sin C = 0$, by (2).
6. $f = \frac{1 - \cos^2 A}{1 - \cos^2 B} = \frac{\sin^2 A}{\sin^2 B} = f^2$; $\therefore l = f$.
7. $f = \frac{a}{\cos A} + \frac{b}{\cos B} = \frac{a \cos B + b \cos A}{\cos A \cos B} = \frac{c}{\cos A \cos B} = l$.
8. $f = (b \cos C + c \cos B) + (c \cos A + a \cos C) + (a \cos B + b \cos A) = l$.
9. $f = \frac{\cos B}{\sin B} + \frac{\cos A}{\sin A} = \frac{\sin A \cos B + \cos A \sin B}{\sin A \sin B} = \frac{\sin(A + B)}{\sin A \sin B}$
 $= \frac{\sin C}{\sin A \sin B} = \frac{c}{b \sin A} = l$.
10. $l = a^2 + b^2 - c^2 + c^2 + a^2 - b^2 + b^2 + c^2 - a^2 = f$.
11. $l = -\frac{b^2 + c^2 - a^2}{2bc} = -\cos A = f$.

CHAPTER XII.

1. $b = a \frac{\sin B}{\sin A} = \sqrt{6} \frac{\sin 45^\circ}{\sin 60^\circ} = \sqrt{6} \times \frac{1}{\sqrt{2}} \times \frac{2}{\sqrt{3}} = 2$.
2. $\sin A = \frac{a}{b} \sin B = \frac{3\sqrt{2}}{2\sqrt{3}} \sin 45^\circ = \frac{\sqrt{3}}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = \frac{\sqrt{3}}{2}$; $\therefore A = 60^\circ$ or 120° .
 The solution is ambiguous, $\therefore a > b$.

$$3. \sin B = \frac{b}{a} \sin A = \frac{2}{\sqrt{2}} \sin 30^\circ = \sqrt{2} \times \frac{1}{2} = \frac{1}{\sqrt{2}}; \therefore B = 45^\circ \text{ or } 135^\circ, \\ \therefore b > a.$$

$$4. \sin B = \frac{b}{a} \sin A = \frac{\sqrt{2}}{2} \sin 45^\circ = \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = \frac{1}{2}; \therefore B = 30^\circ \text{ (only),} \\ \therefore b < a).$$

$$5. a^2 = 3^2 + 5^2 - 2 \cdot 3 \cdot 5 \cdot \cos 120^\circ = 9 + 25 + 30 \times \frac{1}{2} = 49, \therefore a = 7.$$

$$6. \cos C = \left[\left(\frac{3}{2}\right)^2 + \left(\frac{5}{2}\right)^2 - 2^2\right] + (2 \times \frac{3}{2} \times \frac{5}{2}) = \left(\frac{9}{4} + \frac{25}{4} - 4\right) \times \frac{4}{13} = 0; \\ \therefore C = 90^\circ.$$

$$7. \cos A = \frac{(2\sqrt{7})^2 + (3\sqrt{7})^2 - 7^2}{2 \cdot 2\sqrt{7} \cdot 3\sqrt{7}} = \frac{28 + 63 - 49}{84} = \frac{42}{84} = \frac{1}{2}, \therefore A = 60^\circ.$$

$$8. \sin B = \frac{b}{a} \sin A = \frac{5}{2} \times \frac{1}{2} = \frac{5}{4}, \text{ which is impossible. Hence no} \\ \text{triangle can have the parts "given."}$$

$$9. \sin B = \frac{b}{a} \sin A = \frac{\sqrt{20} + 2}{2} \sin 18^\circ = \frac{2(\sqrt{5} + 1)}{2} \times \frac{\sqrt{5} - 1}{4} = \frac{5 - 1}{4} \\ = 1, \therefore B = 90^\circ; \therefore C = 90^\circ - 18^\circ = 72^\circ; \text{ and } c = \sqrt{(b^2 - a^2)} = \\ \sqrt{(24 + 4\sqrt{20} - 4)} = \sqrt{(20 + 8\sqrt{5})} = 2\sqrt{(5 + 2\sqrt{5})}. \quad 90^\circ \text{ and its} \\ \text{supplement are identical.}$$

$$10. \text{Suppose } a, b, C \text{ the given sides and angle; then } A, B, c \text{ are required.}$$

$$a^2 = (3\sqrt{6})^2 + 3^2 (\sqrt{3} + 1)^2 - 2 \cdot 3\sqrt{6} \cdot 3 (\sqrt{3} + 1) \frac{1}{\sqrt{2}} \\ = 54 + 9(4 + 2\sqrt{3}) - 18\sqrt{3}(\sqrt{3} + 1) = 54 + 36 - 54 + 18\sqrt{3} - 18\sqrt{3} \\ = 36; \therefore c = 6. \therefore \sin A = \frac{a}{c} \sin C = \frac{3\sqrt{6}}{6} \frac{1}{\sqrt{2}} = \frac{\sqrt{3}}{2};$$

$$\therefore A = 60^\circ; \therefore B = 180^\circ - (A + C) = 180^\circ - 105^\circ = 75^\circ.$$

$$11. C = \text{greatest and } A = \text{least angle. } \cos C = \frac{3^2 + 4^2 - 5^2}{2 \cdot 3 \cdot 4} = 0, \\ \therefore C = 90^\circ; \therefore \cos A = \frac{3}{4} = .75, \therefore A = 36^\circ 52'.$$

$$12. \text{Let } B \text{ be the angle, } \cos B = \frac{5^2 + 9^2 - 8^2}{2 \cdot 5 \cdot 9} = \frac{42}{90} = .4\bar{6},$$

$$\therefore B = 62^\circ 11' \text{ nearly.}$$

$$13. \cos C = [1^2 + \left(\frac{3}{2}\right)^2 - 2^2] + (2 \cdot 1 \cdot \frac{3}{2}) = (1 + \frac{9}{4} - 4) + \frac{3}{2} = -\frac{1}{4} \times \frac{4}{5} \\ = -\frac{1}{5} = -.65 = -\cos 49^\circ 33' = \cos(180^\circ - 49^\circ 33'); \therefore C = 130^\circ 27'.$$

$$14. c^2 = 5^2 - 3^2 - 2 \cdot 5 \cdot 8 \cdot \cos 97^\circ 53' 50'' = 89 + 80 \cos 82^\circ 6' 10'' \\ = 89 + 80 \times .1375 = 89 + 11 = 100; \therefore c = 10.$$

$$15. a = b \frac{\sin A}{\sin B} = 5\frac{1}{2} \times \frac{\sin 131^\circ 35'}{\sin 30^\circ} \text{ fur.} = \frac{16}{3} \times \frac{\sin 48^\circ 25'}{\sin 30^\circ} \text{ fur.} \\ = 1\frac{2}{3} \times \frac{4}{3} \times \frac{4}{3} \text{ fur.} = 8 \text{ fur.} = 1 \text{ mile.}$$

$$16. \sin A = \frac{a}{b} \sin B = \frac{1}{2} \times \frac{4}{5} = \frac{2}{5} = .75; \therefore A = 48^\circ 25' \text{ or } 131^\circ 35'.$$

17. If a, b contain C , $\therefore \sqrt{(a^2 + ab + b^2)} >$ either a or b , C is greatest angle. $\cos C = \frac{a^2 + b^2 - a^2 - ab - b^2}{2 \cdot a \cdot b} = -\frac{1}{2}$; $\therefore C = 120^\circ$.

$$18. 2f = 2ab \cos C - 2ac \cos B = a^2 + b^2 - c^2 - (a^2 + c^2 - b^2) = 2b^2 - 2c^2 = 2l.$$

$$19. f = a^2 \sin A + b \cdot a \sin B + c \cdot a \sin C = a^2 \sin A + b \cdot b \sin A + c \cdot c \sin A = l.$$

$$20. \frac{2(\cos B \cos C + \cos A)}{2(\cos A \cos C + \cos B)} = \frac{\cos(B+C) + \cos(B-C) - 2\cos(B+C)}{\cos(A+C) + \cos(A-C) - 2\cos(A+C)}$$

$$= \frac{\cos(B-C) - \cos(B+C)}{\cos(A-C) - \cos(A+C)} = \frac{2\sin B \sin C}{2\sin A \sin C} = \frac{b}{a}.$$

Multiplying out, $2f = 2l$, $\therefore f = l$.

$$21. \frac{2(\cos B + \cos C \cos A)}{2(\cos C + \cos A \cos B)} = \frac{-2\cos(C+A) + \cos(C+A) + \cos(C-A)}{-2\cos(A+B) + \cos(A+B) + \cos(A-B)}$$

$$= \frac{\cos(C-A) - \cos(C+A)}{\cos(A-B) - \cos(A+B)} = \frac{2\sin C \sin A}{2\sin A \sin B} = \frac{c}{b}.$$

Multiplying out, $2f = 2l$.

$$22. 2ab \sin(C-30^\circ) - 2bc \sin(A-30^\circ)$$

$$= b[2a(\sin C \frac{1}{2}\sqrt{3} - \cos C \frac{1}{2}) - 2c(\sin A \frac{1}{2}\sqrt{3} - \cos A \frac{1}{2})]$$

$$= b[\sqrt{3}(a \sin C - c \sin A) + c \cos A - a \cos C]$$

$$= \frac{2b}{2}[\sqrt{3} \times 0 + c \cos A - a \cos C] = \frac{1}{2}[2bc \cos A - 2ab \cos C]$$

$$= \frac{1}{2}[b^2 + c^2 - a^2 - (a^2 + b^2 - c^2)] = c^2 - a^2. \text{ By transposition, } f = l.$$

$$23. f = (a \cos C + c \cos A) \cos B + (a \cos B + b \cos A) \cos C = 2a \cos B \cos C$$

$$+ (c \cos B + b \cos C) \cos A = a[\cos(B+C) + \cos(B-C)] + a \cos A$$

$$= -a \cos A + a \cos(B-C) + a \cos A = l.$$

$$24. \text{Add } a \cos A = -a \cos(B+C) \text{ to 23. Then } f = -a \cos(B+C)$$

$$+ a \cos(B-C) = a[\cos(B-C) - \cos(B+C)] = 2a \sin B \sin C.$$

$$25. l = 1 + \frac{2 \sin A \sin B \sin C}{\sin A + \sin B + \sin C}$$

$$= 1 + \frac{2 \cdot 2 \sin \frac{1}{2}A \cos \frac{1}{2}A \cdot 2 \sin \frac{1}{2}B \cos \frac{1}{2}B \cdot 2 \sin \frac{1}{2}C \cos \frac{1}{2}C}{4 \cos \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C} \text{ [p. 64, (4)]}$$

$$= 1 + 4 \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C = f \text{ (p. 66, 33)}.$$

$$26. A+B+C = 180^\circ, \therefore A+B = 180^\circ - C, \therefore \frac{1}{2}(A+B) = 90^\circ - \frac{1}{2}C,$$

$$\therefore f = l.$$

$$27. \text{Remembering that } \frac{1}{2}(A+B) = 90^\circ - \frac{1}{2}C, \frac{a-b}{c} = \frac{\sin A - \sin B}{\sin C}$$

$$= \frac{2 \cos \frac{1}{2}(A+B) \sin \frac{1}{2}(A-B)}{2 \sin \frac{1}{2}C \cos \frac{1}{2}C} = \frac{\sin \frac{1}{2}C \sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}C \cos \frac{1}{2}C} = \frac{\sin \frac{1}{2}(A-B)}{\cos \frac{1}{2}C}.$$

$$28. \frac{a+b}{c} = \frac{\sin A + \sin B}{\sin C} = \frac{2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)}{2 \sin \frac{1}{2}C \cos \frac{1}{2}C} = \frac{\cos \frac{1}{2}(A-B)}{\sin \frac{1}{2}C}.$$

$$\therefore f = l.$$

$$29. l = \frac{\sin B + \sin C}{\sin A} = \frac{2 \sin \frac{1}{2}(B+C) \cos \frac{1}{2}(B-C)}{2 \sin \frac{1}{2}A \cos \frac{1}{2}A} = \frac{\cos \frac{1}{2}(B-C)}{\sin \frac{1}{2}A}.$$

$$= \frac{\cos \frac{1}{2}(C-B)}{\sin^2 A} = f, \text{ for } \cos \frac{1}{2}(C-B) = \sin(90^\circ - \frac{1}{2}C + \frac{1}{2}B)$$

$$= \sin(\frac{1}{2}A + \frac{1}{2}B + \frac{1}{2}C - \frac{1}{2}C + \frac{1}{2}B).$$

30. Let $A = 2a$, then $B = 3a$, $C = 4a$,

$$\therefore l = \frac{\sin 2a + \sin 4a}{2 \sin 3a} = \frac{2 \sin 3a \cos a}{2 \sin 3a} = \cos a = f.$$

$$31. l = \frac{b \cdot b \sin C + c \cdot c \sin B}{b+c} = \frac{b \sin C (b+c)}{b+c} - b \sin C = f.$$

(See fig. to Art. 20.)

$$32. \frac{a^2 - b^2}{c^2} = \frac{\sin^2 A - \sin^2 B}{\sin^2 C} = \frac{\sin(A+B) \sin(A-B)}{\sin^2(A+B)} \\ = \frac{\sin(A-B)}{\sin(A+B)} = \frac{\sin(A-B)}{\sin C}, \therefore f = l.$$

$$33. 2 \sin \frac{1}{2}(A-B) \cdot 2 \sin \frac{1}{2}(B-C) \sin \frac{1}{2}(C-A) \\ = 2 \sin \frac{1}{2}(A-B) [\cos \frac{1}{2}(A+B-2C) - \cos \frac{1}{2}(A-B)] \\ = -2 \sin \frac{1}{2}(B-A) \cos \frac{1}{2}(A+B-2C) - 2 \sin \frac{1}{2}(A-B) \cos \frac{1}{2}(A-B) \\ = -\sin(B-C) - \sin(C-A) - \sin(A-B) = -\frac{b^2 - c^2}{a^2} \sin A \\ = -\frac{c^2 - a^2}{b^2} \sin B - \frac{a^2 - b^2}{c^2} \sin C, \text{ by 32. By transposition, } f = 0.$$

CHAPTER XIII.

- Let $x = \log_2 256$; then $2^x = 256 = 2^8$; $\therefore x = 8$.
Let $x = \log_2 \sqrt{2} 256$; then $(2\sqrt{2})^x = 256$, or $(2^{\frac{3}{2}})^x = 2^8$, or $2^{\frac{3x}{2}} = 2^8$;
 $\therefore \frac{3}{2}x = 8$, $\therefore x = \frac{16}{3} = 5\frac{1}{3}$.
- Let $x = \log_{10} 100$; then $10^x = 100 = 10^2$; $\therefore x = 2$.
Let $x = \log_{10} .001$; then $10^x = .001 = \frac{1}{10^3} = 10^{-3}$; $\therefore x = -3$.
Let $x = \log_{.01} 1000$; then $(.01)^x = 1000$, or $\left(\frac{1}{10^2}\right)^x = 10^3$, or
 $10^{-2x} = 10^3$; $\therefore -2x = 3$, $\therefore x = -\frac{3}{2} = -1\frac{1}{2}$.
- Let $x = \log_3 \sqrt{3} 81\sqrt[3]{3}$; then $(3\sqrt{3})^x = 81\sqrt[3]{3}$, or $(3^{\frac{3}{2}})^x = 3^4 \cdot 3^{\frac{1}{3}}$,
or $3^{\frac{3x}{2}} = 3^{\frac{13}{3}}$; $\therefore \frac{3}{2}x = \frac{13}{3}$; $\therefore x = \frac{26}{9} = 2\frac{8}{9}$.
- Let $x = \log a^{\frac{p}{q}}$ to base $a^{\frac{m}{n}}$; then $(a^{\frac{m}{n}})^x = a^{\frac{p}{q}}$; $\therefore a^{\frac{mx}{n}} = a^{\frac{p}{q}}$,
 $\therefore \frac{mx}{n} = \frac{p}{q}$, $\therefore x = \frac{np}{mq}$.
- Let $x =$ required no.; then $-5 = \log_{10} x$, $\therefore x = 10^{-5} = .00001$.
- Let $x =$ required base; then $2 = \log_x 10$; $\therefore x^2 = 10$,
 $\therefore x = \sqrt{10} = 3.162$.
- Let $x = \log_2 N$, and $y = \log_3 N$; then $N = 2^x$, and $N = 3^y =$
 $(2^x)^y = 2^{xy}$, $\therefore 2^x = 2^{3y}$; $\therefore x = 3y$.

$$8. \text{ Let } x = \log_5 8; \text{ then } 25^x = 8 \text{ or } 5^{2x} = 2^3; \therefore 2x \log 5 = 3 \log 2, \\ \therefore x = \frac{3 \log 2}{2 \log 5} = \frac{3 \log 2}{2(1 - \log 2)} = \frac{.90309}{1.39794} = .646.$$

In 9-22, g stands for given logarithm.

$$9. g = \log x + \log y + \log z.$$

$$10. g = \log(10 \times 2) = \log 10 + \log 2 = 1 + \log 2.$$

$$11. g = \log(3 \times 5 \times 10) = \log 3 + \log 5 + \log 10 = \log 3 + 1 - \log 2 + 1 \\ = 2 - \log 2 + \log 3.$$

$$12. g = \log 1 - \log 2^2 = -2 \log 2.$$

$$13. g = \log(ab^2c^3) - \log(2^3x^4y^5) \\ = \log a + 2 \log b + 3 \log c - 2 \log 2 - 4 \log x - 5 \log y.$$

$$14. g = \log(2 \times 3ab^{\frac{1}{2}}c^{\frac{1}{3}}d^{\frac{1}{4}}) = \log 2 + \log 3 + \log a + \frac{1}{2} \log b + \frac{1}{3} \log c + \frac{1}{4} \log d.$$

$$15. g = \log \frac{(2^4 \times 3)^{\frac{1}{2}}}{3^{\frac{1}{4}}} = \log \frac{2 \cdot 3^{\frac{1}{2}}}{3^{\frac{1}{4}}} = \log \frac{2}{3^{\frac{1}{4}}} = \log 2 - \frac{1}{4} \log 3.$$

$$16. g = \log \frac{\sqrt[3]{3^2 \times 2^3}}{\sqrt[4]{5 \times 20} \sqrt{2^3}} = \log \frac{3^{\frac{2}{3}} \cdot 2^{\frac{3}{3}}}{5^{\frac{1}{4}} \cdot 2^{\frac{3}{2}}} = \log \frac{3^{\frac{2}{3}} \cdot 2^{\frac{1}{3}}}{5^{\frac{1}{4}} \cdot 2^{\frac{3}{2}}} \\ = \log(3^{\frac{2}{3}} \cdot 2^{\frac{1}{3}}) - \frac{1}{4}(1 - \log 2) = \frac{2}{3} \log 2 + \frac{1}{3} \log 2 + \frac{1}{4} \log 3 - \frac{1}{4} \\ = \frac{11}{12} \log 2 + \frac{1}{4} \log 3 - .25.$$

$$17. g = \log(a+b) + \log \sin \frac{1}{2}C + \log \sec \phi.$$

$$18. g = \log c + \log \sin B - \log \sin C.$$

$$19. g = \log(ac \sin B) - \log 2 = \log a + \log c + \log \sin B - \log 2.$$

$$20. g = \log(a-b) + \log \cot \frac{1}{2}C - \log(a+b).$$

$$21. g = \log[(b+c)^{\frac{1}{2}}(b-c)^{\frac{1}{2}}] = \frac{1}{2} \log(b+c) + \frac{1}{2} \log(b-c).$$

$$22. g = \log[(s-b)^{\frac{1}{2}}(s-c)^{\frac{1}{2}}] - \log(b^{\frac{1}{2}}c^{\frac{1}{2}}) \\ = \frac{1}{2} \log(s-b) + \frac{1}{2} \log(s-c) - \frac{1}{2} \log b - \frac{1}{2} \log c.$$

$$23. 2^{2x} = 10^2, \therefore 3x \log 2 = 2 \log 10, \therefore x = \frac{2 \log 10}{3 \log 2} = \frac{2}{.90309} = 2.214.$$

$$24. (x-1) \log 5 = \log 2^2, \therefore x-1 = \frac{2 \log 2}{\log 5} = \frac{2 \log 2}{1 - \log 2} = \frac{.60206}{.69897} \\ = .862; \therefore x = 1.862.$$

$$25. (x-2) \log(10 \times 2) = \log \frac{1}{2^2}, \therefore x-2 = \frac{\log 1 - 2 \log 2}{\log 10 + \log 2} = \frac{-2 \log 2}{1 + \log 2} \\ = -\frac{.60206}{1.30103} = -.462; \therefore x = 1.538.$$

$$26. 2^{x+3} = 25^{3-2x} = 5^{2(3-2x)}, \therefore (x+3) \log 2 = 2(3-2x) \log 5 = \\ (6-4x)(1 - \log 2); \therefore x(4-3 \log 2) = 6-9 \log 2, \text{ or } x(4-.90309) \\ = 6-2.70927, \text{ or } x \times 3.09691 = 3.29073; \therefore x = 1.06.$$

$$27. x(\log 7 - \log 3) = \log 10^2 = 2, \therefore x = \frac{2}{\log 7 - \log 3} = \frac{2}{.36798} = 5.435.$$

$$28. 2^{2x} \cdot (2 \cdot 3)^{2x-5} = 3^{2x-3} \cdot 2^{5x}, \text{ or } 2^{6x-5} \cdot 3^{2x-5} = 3^{2x-3} \cdot 2^{2x}, \\ \therefore 2^{4x-5} \cdot 3^{-3} = 1, \therefore (x-5) \log 2 + (x-3) \log 3 = \log 1 = 0;$$

$$\therefore x(\log 2 + \log 3) = 5 \log 2 + 3 \log 3,$$

$$\therefore x = \frac{5 \log 2 + 3 \log 3}{\log 2 + \log 3} = \frac{1.50515 + 1.43136}{.30103 + .47712} = \frac{2.93651}{.77815} = 3.774.$$

CHAPTER XIV.

1 and 2 answered by inspection.

3. (1) Their significant figures are the same. (2) The first significant figure is equally distant from the units' place and on same side of it. *Otherwise*: (1) $\log 3.56 = \log (.0356 \times 10^2) = 2 + \log .0356$. (2) .0356 and .04 are both between 10^{-2} and 10^{-1} .

4. Answered by inspection.

5. $\overline{3.72045}$	$\overline{2.00243}$	$\overline{1.98628}$	$\overline{.76215}$	6. $\overline{14.32270}$
$\overline{6}$	$\overline{5}$	$\overline{4}$	$\overline{7}$	$\overline{10.01215}$
$\overline{14.32270}$	$\overline{10.01215}$	$\overline{1.94512}$	$\overline{5.33505}$	$\overline{1.94512}$
$\overline{18+4} = \overline{14}$		$\overline{4+3} = \overline{1}$		$\overline{5.33505}$
				$\overline{19.61502}$

7. 4) $\overline{11.02156}$	6) $\overline{2.56821}$	5) $\overline{1.00426}$	3) $\overline{33.82156}$
$\overline{3.25539}$	$\overline{.42804}$	$\overline{1.80085}$	$\overline{11.27385}$
$\overline{11} = \overline{12} + 1$		$\overline{1} = \overline{5} + 4$	

8. $\overline{3.25539}$	$\overline{1.80085}$
$\overline{.42804}$	$\overline{11.27385}$
$\overline{4.82735}$	$\overline{10.52700}$

$$\overline{3} - 1 = \overline{4} \quad \overline{1} + \overline{11} = \overline{1} + 11 = 10.$$

9. $\log 128 = \log 2^7 = 7 \cdot \log 2 = 2.10721$. $125 = 5^3 = 10^2 + 2^3$;

$$\therefore \log 125 = 3 \log 10 - 3 \log 2 = 3 - .90309 = 2.09691.$$

$$2500 = 10^2 \times 5^2 = 10^4 \div 2^2, \therefore \log 2500 = 4 - 2 \log 2 = 3.39794.$$

10. .0005 = $1 + 10^3 \times 2$, $\therefore \log .0005 = -3 - \log 2 = -3.30103 = \overline{4.69897}$.

$$196 = 49 \times 4 = 7^2 \times 2^2, \therefore \log 196 = 2 \log 7 + 2 \log 2 = 1.690196 + .60206 = 2.292256.$$

$$\log (28+5)^{\frac{1}{2}} = \frac{1}{2} \log (56+10) = \frac{1}{2} (\log 7 + 3 \log 2 - 1) = \frac{1}{2} (.845098 + .90309 - 1) = \frac{1}{2} \text{ of } .748188 = .561141.$$

11. $\log \sin 45^\circ = \log 2^{-\frac{1}{2}} = -\frac{1}{2} \log 2 = -.150515 = \overline{1.849485}$.

$$\log \tan 60^\circ = \log 3^{\frac{1}{2}} = \frac{1}{2} \log 3 = .2385607. \quad \log \sec 30^\circ =$$

$$\log (2 \div \sqrt{3}) = \log 2 - \frac{1}{2} \log 3 = .30103 - .2385607 = .0624693.$$

12. $\log \frac{2}{\frac{1}{2}} = \frac{1}{2} (\log 1 - \log 6) = -\frac{1}{2} (\log 2 + \log 3) = -\frac{1}{2} \text{ of } .7781513$

$$= -.2593838 = \overline{1.7406162}. \quad \log (5\frac{1}{2})^{-\frac{1}{2}} = -\frac{1}{2} \log (2\frac{1}{2} \div 3) =$$

$$= -\frac{1}{2} (4 \log 2 - \log 3) = -\frac{1}{2} (1.20412 - .4771213) = -\frac{1}{2} \text{ of } .7269987 =$$

$$= -.3634994 = \overline{1.6365006}.$$

13. $\frac{(2.7)^3 \times (.81)^{\frac{1}{2}}}{(90)^{\frac{1}{2}}} = \frac{3^3 \times 3^{\frac{1}{2}}}{10^3 \times 10^{\frac{1}{2}} \times 3^{\frac{1}{2}} \cdot 10^{\frac{1}{2}}} = \frac{3^{\frac{7}{2}}}{10^{\frac{5}{2}}};$

$$\therefore \text{its } \log = \frac{7}{2} \log 3 - \frac{5}{2} \log 10 = 4.6280766 - 5.85 = \overline{2.7780766}.$$

14. $\log \frac{2^{\frac{1}{2}} \cdot 3^{\frac{1}{2}} \cdot 4}{2^{\frac{1}{2}} \cdot 5^{\frac{1}{2}} \cdot 6} = \log \frac{2^{\frac{1}{2}} \cdot 2^{\frac{1}{2}} \cdot 3^{\frac{1}{2}} \cdot 2^{\frac{1}{2}}}{10^{\frac{1}{2}} \cdot 2^{\frac{1}{2}} \cdot 3^{\frac{1}{2}}} = \log \frac{2^{\frac{1}{2}} \cdot 3^{\frac{1}{2}}}{10^{\frac{1}{2}}} = \frac{1}{2} \log 2 + \frac{1}{2} \log 3 - \frac{1}{2} = .227923 + .05112014 - .16666666 = .1123775.$
15. $\log (3^{20} \times 5^5 + 2^{11}) = \log (3^{20} \times 10^5 + 2^{16}) = 20 \log 3 + 5 - 16 \log 2 = 9.542426 + 5 - 4.81648 = 9.725946, \therefore \text{No. of digits} = 9 + 1 = 10.$
16. $3 = \frac{\sqrt{9} \cdot \sqrt{2}}{\sqrt{2}} = \frac{\sqrt{18} \cdot \sqrt{5}}{\sqrt{10}} = \frac{\sqrt{18} \cdot \sqrt{25}}{\sqrt{10}}; \therefore \log 3 = \frac{1}{2} \log 18 + \frac{1}{2} \log 25 - \frac{1}{2} = .627636 + .349485 - .5 = .477121.$
- $\cdot 16 = \frac{2^4}{100} = \frac{2^4 \cdot 5^4}{10^2 \cdot 25^2} = \frac{10^2}{25^2}; \therefore \log \cdot 16 = 2 - 2 \log 25 = 2 - 2.795880 = \bar{1}.204120. \quad 450 = 18 \times 25, \therefore \log 45 = \log 18 + \log 25 = 2.653212.$

CHAPTER XV.

The logarithms described as "given" are so only as far as regards mantissa.

1. $\log 17257 = 4.2369653$
 $\log 17256 = 4.2369401$
 $\quad \quad \quad \underline{252}$
 $\quad \quad \quad \cdot 74$
 $\quad \quad \quad \underline{10.08}$
 $\quad \quad \quad 176.4$
 pr. pt. for $\cdot 74 = \underline{186}$
 $\log 17256 = 4.2369401$
 $\log 17256 \cdot 74 = 4.2369587$
 $\therefore \log 1725674 = 6.2369587$
2. $4.8924675 = \log 78067$
 $4.8924619 = \log 78066$
 $\quad \quad \quad \underline{56} = \text{diff. for 1,}$
 $4.8924652 = \log \text{ given}$
 $4.8924619 = \log 78066$
 $\quad \quad \quad \underline{33} = \text{pro. part,}$
 $\therefore \text{increase to no.} = \frac{33}{56} = .589,$
 $\text{no. reqd.} = .78066589.$
3. $-1.6784533 = \bar{2}.3215467,$
 $4.3215467 = \log \text{ given}$
 $4.3215363 = \log 20967$
 $\quad \quad \quad \underline{104} = \text{pro. part,}$
 $\text{and } \frac{104}{56} = .5,$
 $\therefore \text{no. reqd.} = .0209675.$
4. $4.2183779 = \log 16534$
 $4.2183517 = \log 16533$
 $\quad \quad \quad \underline{262} = \text{diff. for 1,}$
- $4.2183736 = \text{given log}$
 $4.2183517 = \log 16533$
 $\quad \quad \quad \underline{219} = \text{pro. part,}$
 $\frac{219}{56} = .392857,$
 $\text{no. reqd.} = .00165338359.$
5. Let $x = (.0000083825)^{\frac{1}{2}}$
 $\log x = \frac{1}{2} \log .0000083825$
 $= \frac{1}{2} (6.9233736)$
 $= \bar{2}.3077912,$
 $\log 20313 = 4.3077741$
 $\text{pro. part} = \underline{171},$
 $\log 20314 = 4.3077954$
 $\log 20313 = 4.3077741$
 $\text{diff. for 1} = \underline{213},$
 $\frac{171}{213} = .8,$
 $\therefore x = .0203138.$
6. $\log 105 = \log (\frac{1}{2} \times 3 \times 7)$
 $= 1 - \log 2 + \log 3 + \log 7$
 $= .69897 + .47712 + .84510$
 $= 2.02119,$
 $\therefore \log (1.05)^{20} = 20 \log 1.05$
 $= .02119 \times 20$
 $= .42380$
 $\text{and } \log 2.653 = .42374$
 $\text{pro. part} = \underline{6},$
 $\log 2.654 = .42390$
 $\log 2.653 = .42374$
 $\text{pro. part} = \underline{16},$
 $\frac{16}{56} = .2857,$
 $\therefore (1.05)^{20} = 2.653375.$

$$4. f = 1 - \sqrt{\left(\frac{(s-b)(s-c)}{s(s-a)} \times \frac{(s-a)(s-c)}{s(s-b)}\right)} = 1 - \frac{s-c}{s} = 1 - \frac{2(s-c)}{2s} \\ = \frac{2s-2s+2c}{2s} = l.$$

$$5. f = \frac{a(1-\cos B)}{b(1-\cos A)} = \frac{a}{b} \cdot \frac{2\sin^2 \frac{1}{2}B}{2\sin^2 \frac{1}{2}A} = \frac{a}{b} \cdot \frac{(s-a)(s-c)}{ac} \cdot \frac{bc}{(s-b)(s-c)} \\ = \frac{2(s-a)}{2(s-b)} = l.$$

$$6. f = c \frac{(s-a)(s-c)}{ac} + b \frac{(s-a)(s-b)}{ab} = \frac{(s-a)(s-c) + (s-a)(s-b)}{a} \\ = \frac{(s-a)(s-c+s-b)}{a} = \frac{(s-a)(2s-b-c)}{a} = \frac{(s-a)a}{a} = l.$$

$$7. \therefore \tan \frac{A}{2} \pm \tan \frac{B}{2} = \sqrt{\left\{\frac{(s-b)(s-c)}{s(s-a)}\right\}} \pm \sqrt{\left\{\frac{(s-a)(s-c)}{s(s-b)}\right\}} \\ = \frac{(s-b)\sqrt{s-c} \pm (s-a)\sqrt{s-c}}{\sqrt{[s(s-a)(s-b)]}} \\ \therefore f = \frac{\sqrt{s-c}(s-b+s-a)}{\sqrt{s(s-a)(s-b)}} \times \frac{\sqrt{s(s-a)(s-b)}}{\sqrt{s-c}(s-b-s+a)} = \frac{2s-a-b}{a-b} = l.$$

$$8. \therefore \cot \frac{B}{2} + \cot \frac{C}{2} = \sqrt{\left\{\frac{s(s-b)}{(s-a)(s-c)}\right\}} + \sqrt{\left\{\frac{s(s-c)}{(s-a)(s-b)}\right\}} \\ = \frac{(s-b)\sqrt{s} + (s-c)\sqrt{s}}{\sqrt{[(s-a)(s-b)(s-c)]}} = \frac{(s-b+s-c)\sqrt{s}}{\sqrt{[(s-a)(s-b)(s-c)]}} \\ \therefore f = \sqrt{\left\{\frac{s(s-a)}{(s-b)(s-c)}\right\}} \times \frac{\sqrt{[(s-a)(s-b)(s-c)]}}{a\sqrt{s}} = l.$$

$$9. s = \frac{1}{2}(4+7+9) = 10, \therefore s-b = 3; \therefore \cos \frac{B}{2} = \sqrt{\frac{10 \cdot 3}{4 \cdot 9}} = \frac{1}{2}\sqrt{30}.$$

$$10. s = \frac{1}{2}(\sqrt{3}+1+\sqrt{6}+2) = \frac{1}{2}(3+\sqrt{3}+\sqrt{6}); \\ \therefore s-a = \frac{1}{2}(1-\sqrt{3}+\sqrt{6}); \\ \therefore s-b = \frac{1}{2}(3+\sqrt{3}-\sqrt{6}); \quad s-c = \frac{1}{2}(-1+\sqrt{3}+\sqrt{6}); \\ \therefore \cot \frac{C}{2} = \sqrt{\left\{\frac{(3+\sqrt{3}+\sqrt{6})(-1+\sqrt{3}+\sqrt{6})}{(1-\sqrt{3}+\sqrt{6})(3+\sqrt{3}-\sqrt{6})}\right\}} \\ = \sqrt{\left\{\frac{6\sqrt{2}+2\sqrt{6}+(6+2\sqrt{3})}{6\sqrt{2}+2\sqrt{6}-(6+2\sqrt{3})}\right\}} \\ = \sqrt{\left\{\frac{(6+2\sqrt{3})(\sqrt{2}+1)}{(6+2\sqrt{3})(\sqrt{2}-1)}\right\}} = \sqrt{(\sqrt{2}+1)^2} = \sqrt{2}+1.$$

CHAPTER XVI.

The difference and proportional parts, except in (1), are taken by inspection, when required.

1. Let c, A, B be the given side and angles. Then b is required.
 $C = 180^\circ - (A+B) = 180^\circ - 128^\circ 29' 20'' = 51^\circ 30' 40''$, and $\log b = \log c + L \sin B - L \sin C$.

$$\begin{aligned} L \sin 104^\circ &= L \sin 76^\circ. \\ L \sin 51^\circ 30' &= 9.8935444 \\ \frac{4}{5} \text{ of } 1004 &= 669 \\ L \sin 51^\circ 30' 40'' &= 9.8936113. \end{aligned}$$

$$\begin{aligned} \log c &= 3. \\ L \sin B &= 9.9869041 \\ &12.9869041 \\ L \sin C &= 9.8936113 \\ \log b &= 3.0932928. \\ \log b &= 3.0932928 \\ \log 12396 &= 4.0932816 \\ &112, \end{aligned}$$

$$\begin{aligned} \frac{1}{3} \frac{1}{10} &= .32, \\ \therefore \text{side } b &= 1239.632 \text{ ft.} \end{aligned}$$

$$2. L \sin B = \log b - \log a + L \sin A.$$

$$\begin{aligned} \log b &= 2.8649855 \\ L \sin A &= 9.9785334 \\ &12.8435189 \\ \log a &= 2.9501701 \\ &9.8933488 \\ L \sin 51^\circ 28' &= 9.8933433 \\ &55, \end{aligned}$$

$$\begin{aligned} \frac{46}{1007} \text{ of } 60'' &= 3'', \\ B &= 51^\circ 28' 3''. \end{aligned}$$

3. Suppose a, b, C given. Then

$$\tan \frac{1}{2}(A-B) = \frac{a-b}{a+b} \cot \frac{1}{2}C$$

$$= \frac{16}{180} \cot \frac{1}{2}C = \frac{1}{22} \cot \frac{1}{2}C,$$

$$\therefore L \tan \frac{1}{2}(A-B) = L \cot \frac{1}{2}C - 2 \log 2,$$

$$\begin{aligned} L \cot \frac{1}{2}C &= 10.4101858 \\ 2 \log 2 &= .60206 \end{aligned}$$

$$L \tan \frac{1}{2}(A-B) = 9.8081258$$

$$\begin{aligned} \text{pro. part} &= 429, \\ \text{diff. for } 1' &= 2777, \end{aligned}$$

$$\frac{429}{2777} \text{ of } 60'' = 9'',$$

$$\frac{1}{2}(A-B) = 32^\circ 44' 9''$$

$$\text{and } \frac{1}{2}(A+B) = 68^\circ 45'$$

$$\therefore A = 101^\circ 29' 9''$$

$$B = 36^\circ 0' 51''.$$

$$\begin{aligned} 4. s &= \frac{1}{2}(18+20+22) = 30, \\ s-a &= 12, s-b = 10, s-c = 8. \end{aligned}$$

$$\therefore \tan^2 \frac{1}{2}A = \frac{10 \cdot 8}{30 \cdot 12} = \frac{2}{3^2}$$

$$\begin{aligned} \therefore 2L \tan \frac{1}{2}A &= \log 2 - 2 \log 3 + 20 \\ &= 20.30103 - .9542426 \\ &= 19.3467874, \end{aligned}$$

$$\therefore L \tan \frac{1}{2}A = 9.6733937.$$

$$5. \tan \frac{1}{2}(A-B) = \frac{a-b}{a+b} \cot \frac{1}{2}C = \frac{16}{20} \cot \frac{1}{2}C = \frac{2^3}{10} \cot \frac{1}{2}C;$$

$$\therefore L \tan \frac{1}{2}(A-B) = 3 \log 2 + L \cot 27^\circ 30' - 1 = .90309 + 10.2835233$$

$$- 1 = 10.1866133; \therefore \frac{1}{2}(A-B) = 56^\circ 56' + \frac{2}{3} \frac{44}{100} \text{ of } 60'' =$$

$$56^\circ 56' 51'' .3, \text{ and } \frac{1}{2}(A+B) = 90^\circ - 27^\circ 30' = 62^\circ 30';$$

$$\therefore A = 119^\circ 26' 51'' .3, B = 5^\circ 33' 8'' .7.$$

$$\begin{aligned} 6. s &= \frac{1}{2}(4+5+6) = \frac{15}{2}, s-b = \frac{5}{2}, \cos^2 \frac{1}{2}B = \left(\frac{15}{2} \cdot \frac{5}{2}\right) + (4 \cdot 6) = \frac{85}{2} \\ &= \frac{100}{27}; \therefore 2L \cos \frac{1}{2}B = 2 - 7 \log 2 + 20 = 22 - 2.10721 = 19.89279, \end{aligned}$$

$$\therefore L \cos \frac{1}{2}B = 9.9463950 = L \cos 27^\circ 53' - 90 = L \cos 27^\circ 53' 8'', \text{ for } \frac{8}{100} \text{ of } 60'' = 8''; \therefore B = 55^\circ 46' 16''.$$

$$7. \tan \frac{1}{2}(A-B) = \frac{9-7}{9+7} \cot 32^\circ 6' = \frac{1}{28} \cot 32^\circ 6'; \therefore L \tan \frac{1}{2}(A-B) =$$

$$L \cot 32^\circ 6' - 3 \log 2 = 10.2025255 - .90309 = 9.2994355 =$$

$$L \tan 11^\circ 16' 10'', \text{ for pro. part} = 1139, \text{ and } \frac{1}{1139} \text{ of } 60'' = 10'';$$

$$\therefore A = \frac{1}{2}(A+B) - \frac{1}{2}(A-B) = 69^\circ 10' 10'',$$

$$B = \frac{1}{2}(A+B) + \frac{1}{2}(A-B) = 46^\circ 37' 50''.$$

8. $\tan \frac{1}{2}(A-B) = \frac{9-7}{9+7} \cot \frac{1}{2}C = \frac{1}{23} \cot 23^\circ 42' 30''$, $\therefore L \tan \frac{1}{2}(A-B)$
 $= L \tan 66^\circ 17' 30'' - 3 \log 2 = 10.3573942 - .90309 = 9.4543042$
 $= L \tan 15^\circ 53' 20''$, for pro. part = 1663, and $\frac{1}{23}$ of $60'' = 20''$,
 $\therefore \frac{1}{2}(A-B) = 15^\circ 53' 20''$, and $\frac{1}{2}(A+B) = 66^\circ 17' 30''$;
 $\therefore A = 82^\circ 10' 50''$, $B = 50^\circ 24' 10''$.
9. $s = 12$, $s-a = 5$, &c., $\therefore \tan^2 \frac{1}{2}A = \frac{(s-b)(s-c)}{s(s-a)} = \frac{4.3}{12.5} = \frac{2}{10}$;
 $\therefore L \tan \frac{1}{2}A = \frac{1}{2}(\log 2 - 1 + 20) = \frac{1}{2}(19.30103) = 9.650515$,
 \therefore pro. part = 81, and $\frac{1}{12.5}$ of $10'' = 1''.5$ nearly;
 $\therefore A = (24^\circ 5' 41''.5) \times 2 = 48^\circ 11' 23''$.
 Also, $\tan^2 \frac{1}{2}B = \frac{5.3}{12.4} = \frac{10}{32}$; $\therefore L \tan \frac{1}{2}B = \frac{1}{2}(1 - 5 \log 2 + 20)$
 $= 9.747425$, \therefore pro. part = 67, and $\frac{1}{12.4}$ of $10'' = 1''.5$ nearly;
 $\therefore B = (29^\circ 12' 21''.5) \times 2 = 58^\circ 24' 43''$,
 $\therefore C = 180^\circ - (A+B) = 73^\circ 23' 54''$.
10. Suppose $B = 60^\circ$, $\therefore \tan \frac{1}{2}(A-C) = \frac{19-1}{19+1} \cot 30^\circ = \frac{9\sqrt{3}}{10} = \frac{3\frac{1}{2}}{10}$,
 $\therefore L \tan \frac{1}{2}(A-C) = \frac{1}{2} \log 3 - 1 + 10 = 10.1928032$, $\therefore \frac{1}{2}(A-C) =$
 $57^\circ 19' 11''$, and $\frac{1}{2}(A+C) = 60^\circ$, $\therefore A = 117^\circ 19' 11''$, $B = 2^\circ 40' 49''$.
11. $L \sin B = \log b + L \sin A - \log a = 1.8770256 + 9.7735727 - 1.8080759$
 $= 9.8425224$; and $B = 44^\circ 5' 45''$; and as $b > a$, B also may =
 $135^\circ 54' 15''$.
12. $\log b = \log a + L \sin B - L \sin (B+C) = \log a + L \sin 16^\circ - L \sin 80^\circ$
 $= 2.4048337 + 9.4403381 - 9.9933515 = 1.8518203$,
 \therefore pro. part = 118, and diff. = 611, and $\frac{1}{111} = .193$; $\therefore b = 71.09193$.
13. $\therefore a^2 = c^2 - b^2 = (c+b)(c-b)$, $\therefore 2 \log a = \log (c+b) + \log (c-b)$, \therefore &c.
14. (i.) Obtain A (or B); thus $\tan A = \frac{a}{b}$, $\therefore L \tan A = \log a - \log b + 10$.
 (ii.) $B = 90^\circ - A$. (iii.) $c = a \operatorname{cosec} A$, $\therefore \log c = \log a + L \operatorname{cosec} A - 10$.
15. $\tan A = \frac{9.65}{12.24} = \frac{965}{1224} = \frac{193 \times 5}{153 \times 8} = \frac{193 \times 10}{153 \times 24}$,
 $\therefore L \tan A = 2.2855573 + 1 - 2.1846914 - 1.20412 + 10 = 9.8967459$;
 \therefore pro. part = 343, and diff. = 2598; also $\frac{3.43}{2598}$ of $60'' = 8''$;
 $\therefore A = 38^\circ 15' 8''$, and $B = 51^\circ 44' 52''$.

CHAPTER XVII.

1. Suppose A, B , fig. (13), the cottages, CD the hill; make DE horizontal. Given $AB = 176$ yd., $EDA = 30^\circ$, $EDB = 45^\circ$. Hence $DAC = 30^\circ$, $DBC = 45^\circ$, $ADB = 45^\circ - 30^\circ = 15^\circ$, $\therefore CD = DB \sin 45^\circ$
 $= AB \frac{\sin 30^\circ}{\sin 15^\circ} \frac{1}{\sqrt{2}} = 176 \text{ yd.} \times \frac{1}{2} \times \frac{2\sqrt{2}}{\sqrt{3}-1} \times \frac{1}{\sqrt{2}} = 88(\sqrt{3}+1) \text{ yd.}$
2. In fig. (5), let $CAB = BAD = A$, $BD = a$ ft. Then $BC = BA \sin A$
 $= BD \frac{\sin D \sin A}{\sin DAB} = a \cos CAD = a \cos 2A$.

3. Suppose (13) the fig.; then $AB = BD \frac{\sin 30^\circ}{\sin 15^\circ} = \frac{CD}{\sin 45^\circ} \cdot \frac{\sin 30^\circ}{\sin 15^\circ}$
 $= 50 \text{ ft.} \times \sqrt{2} \times \frac{1}{2} \times \frac{2\sqrt{2}}{\sqrt{3}-1} = 50(\sqrt{3}+1) \text{ ft.}$
4. In fig. (15), $PQH = QHE = 3\alpha$, $QRH = RHE = \alpha$, $QHR = 2\alpha$, $QR = a$;
 $\therefore HP = HQ \sin 3\alpha = QR \frac{\sin \alpha \sin 3\alpha}{\sin 2\alpha} = a \frac{\sin 3\alpha}{2 \cos \alpha}$.
5. In fig. (15), suppose Q, R the two positions of ship; then $QR = 18 \text{ m.}$;
 $\therefore QH = QR \frac{\sin 45^\circ}{\sin 30^\circ} = 18 \text{ m.} \times \frac{1}{\sqrt{2}} \cdot \frac{2}{1} = 18\sqrt{2} \text{ m.}$
6. In fig. (16), $Ss = 14 \text{ m.}$, $LSs = 105^\circ$, $SsL = 45^\circ$, $\therefore SLs = 30^\circ$;
 $\therefore SL = Ss \frac{\sin 45^\circ}{\sin 30^\circ} = 14 \text{ m.} \cdot \frac{1}{\sqrt{2}} \cdot \frac{2}{1} = 14\sqrt{2} \text{ m.}$
 $Ls = 14 \text{ m.} \cdot \frac{\sqrt{3}+1}{2\sqrt{2}} \cdot \frac{2}{1} = 7\sqrt{2}(\sqrt{3}+1) \text{ m.}$
7. In fig. (17), let $PB = 12 \text{ m.}$, $PN = 8 \text{ m.}$, S be the point of meeting;
 then $AP = BP \operatorname{cosec} 45^\circ = 12\sqrt{2} \text{ m.}$; $\therefore AN = (12\sqrt{2} - 8 \text{ m.})$;
 $\therefore NS = AN \frac{\sin 135^\circ}{\sin 30^\circ} = 4\sqrt{2}(3 - \sqrt{2}) \text{ m.} \cdot \frac{1}{\sqrt{2}} \cdot \frac{2}{1} = 8(3 - \sqrt{2}) \text{ m.}$
 $AS = 4\sqrt{2}(3 - \sqrt{2}) \text{ m.} \cdot \frac{\sqrt{3}-1}{2\sqrt{2}} \cdot \frac{2}{1} = 4(3\sqrt{3} - \sqrt{6} - 3 + \sqrt{2}) \text{ m.}$;
 $\therefore BS = 12 + AS = 4(3\sqrt{3} - \sqrt{6} + \sqrt{2}) \text{ m.}$, $PN + NS = 8(4 - \sqrt{2}) \text{ m.}$
8. Suppose (18) the fig., then $AD = AB \sin 60^\circ = BC \frac{\sin 135^\circ}{\sin 15^\circ} \cdot \frac{\sqrt{3}}{2}$
 $= 1 \text{ m.} \cdot \frac{1}{\sqrt{2}} \cdot \frac{2\sqrt{2}}{\sqrt{3}-1} \cdot \frac{\sqrt{3}}{2} = \frac{1}{2}(3 + \sqrt{3}) \text{ m.}$
9. In fig. (19), let AB, CD be the towers, and $ADB = 45^\circ$;
 then $AE = AB = x$ suppose, $\therefore AEB = 45^\circ$, \therefore a circle will pass
 through B, A, E, D ; in which $BDE = 90^\circ$ for $BAE = 90^\circ$. Hence
 $BE^2 = BD^2 + DE^2$ or $(x\sqrt{2})^2 = (x-a)^2 + (x+c)^2 + a^2 + c^2$, for
 $BF = x-a$, and $DF = x+c$; $\therefore 2ax - 2cx = 2(a^2 + c^2)$, $\therefore x = \frac{a^2 + c^2}{a-c}$.
10. Suppose (20) the fig.; then if $CD = x$, $AC = x \cot 30^\circ = x\sqrt{3}$, and
 $BC = x \cot 18^\circ$. But $BAC = 90^\circ$, $\therefore (x\sqrt{3})^2 + a^2 = x^2 \cot^2 18^\circ$
 $= x^2 \frac{5 + \sqrt{5}}{8} \times \frac{16}{6 - 2\sqrt{5}} = x^2 \frac{(5 + \sqrt{5})(6 + 2\sqrt{5})}{8} = x^2(5 + 2\sqrt{5})$,
 $\therefore x^2(2 + 2\sqrt{5}) = a^2$, $\therefore x = \&c$.
11. Suppose (21) the fig.; then $DH = DB = x$ suppose. Now $\tan A =$
 $\cot DCH$, $\therefore \frac{x}{x+a} = \frac{x-b}{x}$, $\therefore x^2 = x^2 + (a-b)x - ab$, $\therefore x = \frac{ab}{a-b}$.
12. Suppose (21) the fig.; then $HCD = 2A$, $HBC = 90^\circ - A$,
 $AHB = 90^\circ - 2A$; $\therefore CD = CH \cos 2A = BH \frac{\sin(90^\circ - A)}{\sin 2A} \cos 2A$
 $= AB \frac{\sin A \cos A \cos 2A}{\sin(90^\circ - 2A) \sin 2A} = \frac{a}{2}$.

13. Take fig. to Art. 136 (3); then $AB = 2y$, $OC = 11y$, $\tan AOB = \frac{1}{10}$;
 $\therefore \frac{\frac{1}{10}}{1 + \frac{1}{10}} = \frac{1}{10}$; $\therefore x^2 + 2x - 99 = 0$ or $(x+11)(x-9) = 0$,
 $\therefore x = 9$.
14. Suppose (22) the fig., and let $\tan DAB = x$, $\therefore x = \tan(BAC + CAD)$
 $= \frac{\frac{1}{2}x + \frac{1}{2}}{1 - \frac{1}{2}x} = \frac{3x+2}{6-x}$, $\therefore x^2 - 3x + 2 = 0$ or $(x-1)(x-2) = 0$,
 $\therefore x = 1$ or 2 .
15. With fig. to Art. 136 (3), let $x =$ breadth reqd.; then $POC = AOB$
 $= AOC - BOC$. $\frac{6}{x} = \left(\frac{230}{x} - \frac{200}{x} \right) + \left(1 + \frac{230 \cdot 200}{x^2} \right) =$
 $\frac{30}{x} \times \frac{x^2}{x^2 + 46000} = \frac{30x}{x^2 + 46000}$, $\therefore 24x^2 = 276000$, $\therefore x^2 = 11500$,
 $\therefore x = 10\sqrt{115}$.
16. Suppose (23) the fig.; then $AB = AD \frac{\sin BDA}{\sin ABD}$
 $= DE \frac{\sin E}{\sin EAD} \cdot \frac{\sin(\beta - \alpha)}{\sin \beta} = a \frac{\cos \theta}{\sin(\theta - \alpha)} \cdot \frac{\sin(\beta - \alpha)}{\sin \beta}$.
17. Suppose (24) the fig., and let $BC = x$, then $AC = x \cot \alpha$, $DC = x \cot \beta$,
and $\therefore CAD = 90^\circ$, $\therefore a^2 = x^2 \cot^2 \beta - x^2 \cot^2 \alpha = x^2 (\operatorname{cosec}^2 \beta - \operatorname{cosec}^2 \alpha)$
 $= x^2 \frac{\sin^2 \alpha - \sin^2 \beta}{\sin^2 \alpha \sin^2 \beta} = x^2 \frac{\sin(\alpha + \beta) \sin(\alpha - \beta)}{\sin^2 \alpha \sin^2 \beta}$,
 $\therefore x = \frac{a \sin \alpha \sin \beta}{\sqrt{\sin(\alpha + \beta) \sin(\alpha - \beta)}}$.
18. Construct as in (25); and let $CH = x$, $BP = y$. By Geometry,
since $CAH = CBH$, a circle will pass round $ABCH$, and therefore
 $y(y+20) = 20(x+20)$ (1).
Also $1 = \tan CBH = \left(\frac{x+20}{y} - \frac{20}{y} \right) + \left(1 + \frac{(x+20)20}{y^2} \right)$
 $= \frac{xy}{y^2 + (x+20)20} = \frac{xy}{y^2 + y(y+20)}$ [by (1)] $= \frac{x}{2y+20}$;
 $\therefore 2y+20 = x$, $\therefore 2(y+20) = x+20$ (2).
From (1) & (2), $y = 40$, $\therefore x = 100$.
19. Construct as in (26), and let $CD = a$, $BC = AD = y$;
 $\therefore x = DE - CE = y(\sin A - \sin B) = y \cdot 2 \sin \frac{1}{2}(A-B) \cos \frac{1}{2}(A+B)$ (1),
and $a = BE - AE = y(\cos B - \cos A) = y \cdot 2 \sin \frac{1}{2}(A-B) \sin \frac{1}{2}(A+B)$ (2).
Dividing (1) by (2), $x = a \cot \frac{1}{2}(A+B)$.
20. Given, in triangle ABC , $c = 100$ yds., $A = 53^\circ 20'$, $B = 59^\circ 30'$;
reqd. b . $C = 180^\circ - (A+B) = 67^\circ 10'$, $\sin C = \cos 22^\circ 50'$; $\log b =$
 $\log c + L \sin b - L \sin C = 2 + 9 \cdot 9353204 - 9 \cdot 9645602 = 1 \cdot 9707602$;
 $\therefore b = 93 \cdot 489$ yds.
21. Standing at the extremity of the shadow, the sun will be seen in a
line with the top of the tower. Its angular altitude is therefore
that of the top of the tower. Suppose it $= \alpha$. Then $\tan \alpha = \frac{1 \frac{1}{2}}{2}$,
 $\therefore L \tan \alpha = \log 2 + 10 = 10 \cdot 3010300$. Hence pro. part $= 306$,
diff. for $60'' = 3159$, $\frac{3159}{3010300}$ of $60'' = 6''$, $\therefore \alpha = 63^\circ 26' 6''$.

22. Given (fig. 11) $DA = 30$ ft., $BAC = 51^\circ$, $BDC = 46^\circ$; reqd. AC ,
 $= x$ feet, suppose.

$$ABD = BAC - D = 5^\circ.$$

$$\log x = \log 30 + L \sin 46^\circ + \bar{L} \cos 51^\circ - L \sin 5^\circ - 10$$

$$x = AB \cos 51^\circ = AD \frac{\sin 46^\circ}{\sin 5^\circ} \cos 51^\circ,$$

$$= 1.47712$$

$$+ 9.85693$$

$$+ 9.79887 - 18.94029$$

$$\text{and } \cos 51^\circ = \sin 39^\circ.$$

$$\log 30 = \log 3 + \log 10 = 1.47712.$$

$$= 21.13292 - 18.94029$$

$$x = 155.8.$$

$$= 2.19263.$$

23. Given (fig. 11) $DA = 600$ yds.

$$D = 30^\circ, BAC = 60^\circ; \text{ reqd. } BC.$$

$$ABD = BAC - D = 30^\circ = D,$$

$$\therefore AB = AD,$$

$$\therefore BC = AB \sin 60^\circ$$

$$= 600 \times \frac{1}{2} \sqrt{3} = 300 \sqrt{3}.$$

$$\log BC = \log 3 + \log 100 + \frac{1}{2} \log 3$$

$$= 2.47712$$

$$+ .23856$$

$$= 2.71568.$$

$$\text{pro. pt.} = 811, \text{ diff.} = 843, \frac{811}{843} = .96,$$

$$BC = 519.6.$$

24. Given a, b, C ; reqd. A, B .

$$\tan \frac{1}{2}(A-B) = \frac{a-b}{a+b} \cot \frac{1}{2}C$$

$$L \tan \frac{1}{2}(A-B) = L \cot \frac{1}{2}C - \log 3$$

$$= 10.4771213$$

$$= \frac{21-10\frac{1}{2}}{21+10\frac{1}{2}} \cot \frac{1}{2}C = \frac{1}{2} \cot \frac{1}{2}C.$$

$$= .4771213$$

$$= 10,$$

$$\log 3 = \log 3.0$$

$$\therefore \log \tan \frac{1}{2}(A-B) = 0 = \log 1.$$

$$= \log 15 + \log 2 - \log 10 = .4771213.$$

$$\therefore \frac{1}{2}(A+B) = 90^\circ - \frac{1}{2}C = 71^\circ 33' 54'', \frac{1}{2}(A-B) = 45^\circ; \therefore \text{adding and subtracting, } A = 116^\circ 33' 54'', B = 26^\circ 33' 54''.$$

25. Given (fig. 27) $AP = 220$ ft., $BAC = 71^\circ$, $BPC = 55^\circ$; reqd. BC, BA .

$$BC = AC \sin CAB$$

$$BA = AC \cos CAB$$

$$= AP \frac{\sin CPA}{\sin ACP} \sin CAB$$

$$= AP \frac{\sin CPA}{\sin ACB} \cos CAB$$

$$\log BC = \log AP + L \sin 55^\circ$$

$$\log BA = \log AP + L \sin 55^\circ$$

$$+ L \sin 71^\circ - L \sin 16^\circ - 10$$

$$+ L \cos 71^\circ - L \sin 16^\circ - 10$$

$$= 2.34242$$

$$= 2.34242$$

$$+ 9.91336$$

$$+ 9.91336$$

$$+ 9.97567 - 19.44034$$

$$+ 9.51264 - 19.44034$$

$$= 22.23145 - 19.44034$$

$$= 21.76842 - 19.44034$$

$$= 2.79111;$$

$$= 2.32808;$$

$$\therefore BC = 618.17 \text{ ft.}$$

$$\therefore BA = 212.85 \text{ ft.}$$

26. Given (fig. 28) $BD = 180$ yds., $BDP = 41^\circ$, $BDQ = 68^\circ$.

$$DP = BD \sec BDP,$$

$$DQ = BD \sec BDQ,$$

$$\log DP = \log 180 - L \cos 41^\circ + 10$$

$$\log DQ = \log 180 - L \cos 68^\circ + 10$$

$$= 12.255273$$

$$= 12.255273$$

$$- 9.877780$$

$$- 9.573575$$

$$= 2.377493;$$

$$= 2.681698;$$

$$\therefore BP = 238.5 \text{ yds.}$$

$$\therefore BQ = 480.5 \text{ yds.}$$

27. Given (fig. 29) $CD = 64$ ft., $BDC = 80^\circ = BCD$, $ACB = 30^\circ$, $BAC = 20^\circ$; reqd. AB . $AB = BC \frac{\sin 30^\circ}{\sin 20^\circ} = CD \frac{\sin 80^\circ \sin 30^\circ}{\sin 20^\circ \sin 20^\circ}$
 $= 64 \frac{4 \cos 40^\circ \cdot \sin 20^\circ \cos 20^\circ}{\sin 20^\circ \sin 20^\circ} \frac{1}{2} = 128 \cdot \cos 40^\circ \cot 20^\circ$,
 $\therefore \log AB = 7 \log 2 + L \cos 40^\circ - L \tan 20^\circ = 2.1072100 + 9.8842540$
 $- 9.5610659 = 11.9914640 - 9.5610659 = 2.4303981$,
 $\therefore AB = 296.40031$.
28. The formula $\cot \frac{1}{2}C = \frac{a+b}{a-b} \tan \frac{1}{2}(A-B)$ will determine $\frac{1}{2}C$; then $\frac{1}{2}(A+B) = 90^\circ - \frac{1}{2}C$, and is known; and $A = \frac{1}{2}(A+B) + \frac{1}{2}(A-B)$, $B = \frac{1}{2}(A+B) - \frac{1}{2}(A-B)$, and are both known. Side c follows from $c = a \frac{\sin C}{\sin A}$.
29. $C = 180^\circ - (A+B)$; then $a-b$ is given by $a-b = (a+b) \frac{\tan \frac{1}{2}(A-B)}{\cot \frac{1}{2}C}$. Sides a, b follow, and the solution is in ordinary course.
30. $\sin \frac{1}{2}(A-B) = \frac{a-b}{c} \cos \frac{1}{2}C$ gives $\frac{1}{2}(A-B)$, and (Euc. I. 32) $\frac{1}{2}(A+B)$; after which a and b follow, and then c , in ordinary way.
31. $\frac{1}{2}(b+c-a) = s-a$, $\frac{1}{2}[b-(c-a)] = s-c$; hence $\tan \frac{1}{2}C = \frac{s-a}{s-c} \tan \frac{1}{2}A$ will give C , and the remaining solution follows.
32. $\frac{1}{2}(b+c+a) = s$, $\frac{1}{2}(c+a-b) = s-b$; then $\cot \frac{1}{2}C = \frac{s}{s-b} \tan \frac{1}{2}A$ gives C , and the rest follow as usual.

CHAPTER XVIII.

1. $f = \sqrt{\left(\frac{\Delta}{s} \cdot \frac{\Delta}{s-a} \cdot \frac{\Delta}{s-b} \cdot \frac{\Delta}{s-c} \right)} = \frac{\Delta^2}{\Delta} = l$.
2. $l = \frac{1}{4} \frac{abc}{\Delta} \cdot \frac{\Delta}{s} \cdot s = f$. 3. $f = \frac{bc}{2s} \cdot \frac{2\Delta}{bc} = \frac{\Delta}{s} = l$.
4. $f = \frac{2\Delta}{bc} \cdot \frac{bc}{2(s-a)} = \frac{\Delta}{s-a} = l$.
5. $l = \sqrt{\left\{ \frac{(s-b)(s-c)}{bc} \right\}} \times \sqrt{[bc \cdot s(s-a)]} = \sqrt{[s(s-a)(s-b)(s-c)]} = f$.
6. $l = \frac{\Delta}{s} \cdot \frac{\Delta}{s-a} \cdot \frac{s-b}{\Delta} \cdot \frac{s-c}{\Delta} = \frac{(s-b)(s-c)}{s(s-a)} = f$.
7. $a \sin \frac{1}{2}B \sin \frac{1}{2}C \sec \frac{1}{2}A = a \sqrt{\left\{ \frac{(s-a)(s-c)}{ac} \cdot \frac{(s-a)(s-b)}{ab} \cdot \frac{bc}{s(s-a)} \right\}}$
 $= a \sqrt{\left\{ \frac{(s-a)(s-b)(s-c)}{a^2 \cdot s} \right\}} = \frac{\Delta}{s} = r$.

- $$\begin{aligned} b \sin \frac{1}{2} C \sin \frac{1}{2} A \sec \frac{1}{2} B &= b \sqrt{\left\{ \frac{(s-a)(s-b)}{ab} \cdot \frac{(s-b)(s-c)}{bc} \cdot \frac{ac}{s(s-b)} \right\}} \\ &= b \times \frac{1}{b} \sqrt{\left\{ \frac{(s-a)(s-b)(s-c)}{s} \right\}} = r. \\ c \sin \frac{1}{2} A \sin \frac{1}{2} B \sec \frac{1}{2} C &= c \sqrt{\left\{ \frac{(s-b)(s-c)}{bc} \cdot \frac{(s-a)(s-c)}{ac} \cdot \frac{ab}{s(s-c)} \right\}} \\ &= c \times \frac{1}{c} \sqrt{\left\{ \frac{(s-a)(s-b)(s-c)}{s} \right\}} = r. \\ 8. a \cos \frac{1}{2} B \cos \frac{1}{2} C \sec \frac{1}{2} A &= a \sqrt{\left\{ \frac{s(s-b)}{ac} \cdot \frac{s(s-c)}{ab} \cdot \frac{bc}{s(s-a)} \right\}} \\ &= a \times \frac{1}{a} \sqrt{\left\{ \frac{s(s-b)(s-c)}{s-a} \right\}} = r_a. \\ 9. l = s \cdot c + \left(\frac{\cos \frac{1}{2} A}{\sin \frac{1}{2} A} + \frac{\cos \frac{1}{2} B}{\sin \frac{1}{2} B} \right) &= s \cdot c \times \frac{\sin \frac{1}{2} A \sin \frac{1}{2} B}{\cos \frac{1}{2} A \sin \frac{1}{2} B + \cos \frac{1}{2} B \sin \frac{1}{2} A} \\ &= \frac{s \cdot c \cdot \sin \frac{1}{2} A \sin \frac{1}{2} B}{\sin \frac{1}{2} (A+B)} = s \cdot c \frac{\sin \frac{1}{2} A \sin \frac{1}{2} B}{\cos \frac{1}{2} C} = s \cdot r [\sec (7)] = f. \\ 10. l = \frac{a^2}{2} \left(1 - \frac{b^2}{a^2} \right) \frac{\sin A \sin B}{\sin (A-B)} &= \frac{a^2}{2} \left(1 - \frac{\sin^2 B}{\sin^2 A} \right) \frac{\sin A \sin B}{\sin (A-B)} \\ &= \frac{a^2}{2} \frac{\sin^2 A - \sin^2 B}{\sin^2 A} \frac{\sin A \sin B}{\sin (A-B)} = \frac{a^2}{2} \frac{\sin (A+B) \sin (A-B)}{\sin^2 A} \frac{\sin A \sin B}{\sin (A-B)} \\ &= \frac{a^2}{2} \frac{\sin^2 A}{\sin A} \frac{\sin B \sin C}{\sin A} = \frac{a^2}{2} \frac{\sin B \sin C}{\sin A} = f. \\ 11. l = 2s \sqrt{\left\{ \frac{bc}{s(s-a)} \cdot \frac{ac}{s(s-b)} \cdot \frac{ab}{s(s-c)} \right\}} \\ &= 2s \cdot \frac{abc}{s} \sqrt{\left\{ \frac{1}{s(s-a)(s-b)(s-c)} \right\}} = 8 \frac{abc}{4\Delta} = f. \\ 12. l = s^2 \sqrt{\left\{ \frac{(s-b)(s-c)(s-a)(s-a)(s-b)}{s(s-a) \cdot s(s-b) \cdot s(s-c)} \right\}} \\ &= \frac{s^2}{s^2} \sqrt{[s(s-a)(s-b)(s-c)]} = f. \\ 13. l = \frac{1}{2} r (2R \sin A + 2R \sin B + 2R \sin C) &= \frac{1}{2} r (a+b+c) = \frac{\Delta}{s} \cdot s = f. \\ 14. f = 4 \frac{abc}{4\Delta} \sqrt{\left\{ \frac{(s-b)(s-c)(s-c)(s-a)(s-a)(s-b)}{bc \cdot ca \cdot ab} \right\}} \\ &= \frac{1}{\Delta} \cdot \frac{\Delta^2}{s} = \frac{\Delta}{s} = l. \\ 15. f = 4 \frac{abc}{4\Delta} \sqrt{\left\{ \frac{(s-b)(s-c) \cdot s(s-b) \cdot s(s-c)}{bc \cdot ca \cdot ab} \right\}} &= \frac{1}{\Delta} \cdot \frac{\Delta^2}{s-a} = \frac{\Delta}{s-a} = l. \\ 16. f = \frac{s(s-a) + s(s-b) + s(s-c)}{s(s-a)(s-b)(s-c)} &+ \frac{s(s-a)(s-b)(s-c)}{(s-b)(s-c)} + \frac{s(s-a)(s-b)(s-c)}{(s-c)(s-a)} + \frac{s(s-a)(s-b)(s-c)}{(s-a)(s-b)} \\ &= \frac{\Delta}{s-b} \cdot \frac{\Delta}{s-c} + \frac{\Delta}{s-c} \cdot \frac{\Delta}{s-a} + \frac{\Delta}{s-a} \cdot \frac{\Delta}{s-b} = l. \end{aligned}$$

17. $4 \Delta \cot A = 4 \cdot \frac{1}{2} bc \sin A \cot A = 2bc \cos A = b^2 + c^2 - a^2$;
 $\therefore 4 \Delta (\cot A + \cot B + \cot C) = b^2 + c^2 - a^2 + c^2 + a^2 - b^2 + a^2 + b^2 - c^2$
 $= a^2 + b^2 + c^2$; $\therefore f = l$.
18. By XII., Ex. 24, $f = 2a \sin B \sin C = 2R \sin A \times 2 \sin B \sin C = l$.
19. $f = -\frac{s}{\Delta} + \frac{s-a}{\Delta} + \frac{s-b}{\Delta} + \frac{s-c}{\Delta} = \frac{-s+s-a+s-b+s-c}{\Delta}$
 $= \frac{2s-(a+b+c)}{\Delta} = l$.
20. $f = \Delta \left(-\frac{1}{s} + \frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} \right) = \Delta \left(\frac{-s+a+s}{s(s-a)} + \frac{s-c+s-b}{(s-b)(s-c)} \right)$
 $= \Delta a \left\{ \frac{1}{s(s-a)} + \frac{1}{(s-b)(s-c)} \right\} = \Delta a \frac{s^2 - (b+c)s + bc + s^2 - as}{s(s-a)(s-b)(s-c)}$
 $= \frac{a}{\Delta} [2s^2 + bc - (b+c+a)s] = \frac{abc}{\Delta} = l$.
21. $R+r = \frac{c}{2 \sin C} + \frac{2\Delta}{2s} = \frac{c}{2} + \frac{ab}{a+b+c} = \frac{ca+cb+c^2+2ab}{2(a+b+c)}$
 $= \frac{c(a+b)+a^2+b^2+2ab}{2(a+b+c)} = \frac{(a+b)c+(a+b)^2}{2(a+b+c)} = \frac{(a+b)(c+a+b)}{2(a+b+c)} = \frac{a+b}{2}$.
22. $s(s-c) = \frac{(a+b+c)(a+b-c)}{2 \cdot 2} = \frac{(a+b)^2 - c^2}{4} = \frac{a^2+b^2+2ab-c^2}{4}$
 $= \frac{1}{2} ab = \Delta$.
23. $\therefore s = \frac{1}{2}(7+8+9) = 12$, $\therefore \Delta = \sqrt{(12 \cdot 5 \cdot 4 \cdot 3)} = 26.83$.
24. $\sin B = \frac{b}{a} \sin A = \frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{2}}$, $\therefore B = 45^\circ$ and $C = 75^\circ$;
 $\therefore \Delta ABC = \frac{1}{2} ab \sin C = \frac{1}{2} \sqrt{3} \sqrt{2} \cdot \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{1}{4} (3+\sqrt{3})$.
25. $\Delta = \frac{1}{2} 20 \cdot 20 \cdot \frac{1}{2} \sqrt{3} = 100 \sqrt{3} = 100 \times 1.732 = 173.2$.
26. Area of first triangle $= \frac{1}{2} \cdot 3 \cdot 12 \cdot \frac{1}{2} = 9$. Hence, if $x = \text{hyp. reqd.}$,
 by (3), p. 109, $\frac{1}{2} x^2 \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \cdot 1 = 9$, $\therefore x^2 = 36$, $\therefore x = 6$.
27. By Euc. I. 34, reqd. area $= 2 \cdot \frac{1}{2} ab \sin A = ab \sin A$.
28. By expression just found, areas are as
 $20 \cdot 30 \cdot \sin 60^\circ : 30 \cdot 40 \cdot \sin 120^\circ = 20 \cdot 30 : 30 \cdot 40 = 1 : 2$.
29. Let d be cut at intersection of diagonals into the parts u, v , and d' into x, y . Then, remembering (2), p. 50, area of the quadrilateral
 $= \frac{1}{2} (ux+xv+vy+yu) \sin \alpha = \frac{1}{2} (u+v)(x+y) \sin \alpha = \frac{1}{2} dd' \sin \alpha$.
30. Suppose CD drawn, in fig. 2, p. 73, bisecting BB' at right angles.
 Then, evidently, $\triangle CDB = \triangle CDB'$, and $\therefore \triangle CAB + \triangle CAB' = 2 \triangle CAD = AD \cdot DC = b \cos A \cdot b \sin A$.
31. In fig. p. 110, $AO = OF \operatorname{cosec} \frac{1}{2} A = \frac{2\Delta}{2s \cdot \sin \frac{1}{2} A} = \frac{bc \sin A}{2s \cdot \sin \frac{1}{2} A} = l$.
32. $\therefore \frac{a}{2 \sin A} = R$, $\therefore a = 2R \sin A$. Similarly, $b = 2R \sin B$, $c = 2R \sin C$.

CHAPTER XIX.

1. Reqd. circumf. = $2\pi \cdot 2$ ft. = $3 \cdot 1416 \times 4$ ft. = $12 \cdot 5664$ ft.
2. Area of circle = $\pi \cdot 35^2$ sq. yds.; area of inner ring = $\pi \cdot 21^2$ sq. yds.;
 \therefore area of accommodation = $\pi (35^2 - 21^2)$ sq. yds. =
 $\pi (35 + 21)(35 - 21)$ sq. yds. = $\frac{\pi}{4} \times 56 \times 14$ sq. yds. = 2464 sq. yds.;
 \therefore reqd. no. = $\frac{1}{4}$ of $2464 = 1232$.
3. Let R be radius; then $1 = 2R \sin \frac{1}{2}\pi = 2R \times \frac{1}{2\sqrt{2}} \sqrt{(5 - \sqrt{5})}$,
 (p. 61, Ex. 41); $\therefore R = \frac{\sqrt{2}}{\sqrt{(5 - \sqrt{5})}} = \frac{\sqrt{2} \sqrt{(5 + \sqrt{5})}}{\sqrt{20}} = \sqrt{\left(\frac{5 + \sqrt{5}}{10}\right)}$.
4. $2f = a \left(\operatorname{cosec} \frac{\pi}{n} + \cot \frac{\pi}{n} \right) = a \frac{1 + \cos \frac{\pi}{n}}{\sin \frac{\pi}{n}} = a \frac{2 \cos^2 \frac{\pi}{2n}}{2 \sin \frac{\pi}{2n} \cos \frac{\pi}{2n}} = 2l$.
5. Area = $\pi r^2 = \frac{1}{2} \cdot 2\pi r \cdot r = \frac{1}{2}$ circumf. \times rad.
6. Area = $\frac{1}{2}\pi (2r)^2 = \frac{1}{2}$ of $3 \cdot 1416 \times (2r)^2 = (\text{diameter})^2 \times \cdot 7854$.
7. Area = $\frac{1}{2}\theta r^2 = \frac{1}{2}r \times \theta r = \frac{1}{2}r \cdot \text{arc}$. [See (2), p. 115.]
8. Area of segment = that of sector - that of triangle between chord and radii to its extremities = $\frac{1}{2}\theta r^2 - \frac{1}{2}r \cdot r \sin \theta = \frac{1}{2}r^2 (\theta - \sin \theta)$.
9. On making the figure, it is evident that, R, r being the radii, and t the tangent, $\theta = R^2 - r^2$. Evidently, area between circumferences = $\pi R^2 - \pi r^2 = \pi (R^2 - r^2) = \pi t^2$.
10. If ρ be radius of circle, area of inscribed \times area of circumscribed equilateral triangle = $\frac{3}{4}\rho^2 \sin 120^\circ \times 3\rho^2 \tan 60^\circ = \frac{3}{4}\rho^4$.
 Also, (area of hexagon) $^2 = (3\rho^2 \sin 60^\circ)^2 = \frac{3}{4}\rho^4$.
11. Let α, β, γ be the sides of pentagon, hexagon, and decagon respectively, and ρ radius; then $\beta^2 + \gamma^2 = (2\rho \sin \frac{1}{6}\pi)^2 + (2\rho \sin \frac{1}{10}\pi)^2 =$
 $4\rho^2 \left(\frac{1}{4} + \frac{6 - 2\sqrt{5}}{16} \right) = 4\rho^2 \frac{10 - 2\sqrt{5}}{16} = 4\rho^2 \frac{5 - \sqrt{5}}{8} = (2\rho \sin \frac{1}{4}\pi)^2 = \alpha^2$.
12. Suppose α the angle subtended at centre by the side a ; then radius = ma ; $\therefore a = 2ma \sin \frac{1}{2}\alpha$, $\therefore \cos \alpha = 1 - 2 \sin^2 \frac{1}{2}\alpha = 1 - \frac{2}{4m^2}$
 $= \frac{2m^2 - 1}{2m^2}$, $\therefore \sec \alpha = \frac{2m^2}{2m^2 - 1}$.
13. Let x and $x+1$ be required numbers. By Euc. I. 32, Cor. 1,
 $2x \cdot 90^\circ - 360^\circ = \frac{2(x+1)90^\circ - 360^\circ}{4}$, $\therefore 45x - 90 = \frac{45x - 45}{4} - 1$,
 $\therefore 45(x^2 - x - 2) = 45x^2 - 45x - x^2 - x$, $\therefore x^2 + x - 90 = 0$, $\therefore x = 9$
 and $x+1 = 10$.
14. Let ρ, ρ' be the radii; then, by question, $\frac{3}{4}\pi\rho = 2\pi\rho'$, $\therefore \rho : \rho' = 3 : 1$.
 Let A be no. of degrees reqd., and θ the corresponding circ. meas.;
 then $\theta = \frac{\rho}{\rho'} A = 3$, $\therefore A : 3 = 180^\circ : \pi$, $\therefore A = \frac{180^\circ \times 3}{\pi} = \frac{225000^\circ}{1309}$
 $= 171^\circ 53' 14''$.

15. The three radii are as $1\frac{1}{2} : 2\frac{1}{2} : 1\frac{3}{4} + 2\frac{1}{4} = 10 : 15 : 25 = 2 : 3 : 5$.

Let them be 2ρ , 3ρ , 5ρ respectively; then the areas are as

$$\pi \cdot 4\rho^2 : \pi \cdot 9\rho^2 : \pi \cdot 25\rho^2 = 4 : 9 : 25.$$

16. Let x = no. of sides, and ρ radius of circle; then, by question,

$$\frac{1}{2} x \rho^2 \sin \frac{2\pi}{x} : x \rho^2 \tan \frac{\pi}{x} = 3 : 4, \therefore 2 \sin \frac{2\pi}{x} = 3 \tan \frac{\pi}{x},$$

$$\therefore 4 \sin \frac{\pi}{x} \cos \frac{\pi}{x} = 3 \sin \frac{\pi}{x}, \therefore \cos \frac{\pi}{x} = \frac{\sqrt{3}}{2} = \cos \frac{\pi}{6}, \therefore x = 6.$$

CHAPTER XX.

1. $\tan \alpha = \frac{\sin \alpha}{\sqrt{(1-\sin^2 \alpha)}} = \frac{s}{\sqrt{(1-s^2)}}.$
2. $\operatorname{cosec} \alpha = \frac{1}{\sqrt{(1-\cos^2 \alpha)}} = \frac{1}{\sqrt{(1-c^2)}}.$
3. $\operatorname{vers} \alpha = 1 - \frac{\cos \alpha}{\sin \alpha} \sin \alpha = 1 - \frac{\cot \alpha}{\sqrt{(\cot^2 \alpha + 1)}} = 1 - \frac{t}{\sqrt{(t^2 + 1)}} = \frac{\sqrt{(t^2 + 1)} - t}{\sqrt{(t^2 + 1)}}.$
4. If $\alpha = \cos^{-1} x$, or $\cos \alpha = x$, then $\sec \alpha = \frac{1}{x}$, or $\alpha = \sec^{-1} \frac{1}{x}.$
5. If $\alpha = \sin^{-1} x$, or $\sin \alpha = x$, then $\cos \alpha = \sqrt{(1-x^2)}$, or $\alpha = \cos^{-1} \sqrt{(1-x^2)}.$
6. If $\alpha = \cot^{-1} x$ or $\cot \alpha = x$, then $\operatorname{cosec} \alpha = \sqrt{(x^2 + 1)}$, or $\alpha = \operatorname{cosec}^{-1} \sqrt{(x^2 + 1)}.$
7. Let $\alpha = \sec^{-1} x$, or $\sec \alpha = x$, then $\tan \alpha = \sqrt{(x^2 - 1)}.$
8. Let $\alpha = \tan^{-1} \frac{b}{a}$, or $\tan \alpha = \frac{b}{a}$, then $a \sec \alpha = a \sqrt{\left(\frac{b^2}{a^2} + 1\right)} = \sqrt{(a^2 + b^2)}.$
9. Let $\alpha = \sin^{-1} \frac{b}{a}$, or $\sin \alpha = \frac{b}{a}$, then $a \cos \alpha = a \sqrt{\left(1 - \frac{b^2}{a^2}\right)} = \sqrt{(a^2 - b^2)}.$
10. Let $\alpha = \sin^{-1} \sqrt{\left[\frac{1}{2}(1-x)\right]}$, or $\sin \alpha = \sqrt{\left[\frac{1}{2}(1-x)\right]}$, then $\cos 2\alpha = 1 - 2 \cdot \frac{1}{2}(1-x) = x$, or $2\alpha = \cos^{-1} x.$
11. Let $\alpha = \sin^{-1} x$, or $\sin \alpha = x$; then $\sin 2\alpha = 2 \sin \alpha \cos \alpha = 2x \sqrt{(1-x^2)}.$
12. Let $\alpha = \tan^{-1} \sqrt{\frac{x}{a}}$, or $\tan \alpha = \sqrt{\frac{x}{a}}$; $\cos 2\alpha = 1 - 2 \frac{\sin^2 \alpha}{\cos^2 \alpha} = 1 - 2 \frac{\frac{x}{a}}{1 + \frac{x}{a}} = \frac{1-x}{1+\frac{x}{a}} = \frac{a-x}{a+x}$, or $2\alpha = \cos^{-1} \frac{a-x}{a+x}.$
13. $2 \tan^{-1} \frac{1}{2} = \tan^{-1} \frac{2 \cdot \frac{1}{2}}{1 - \frac{1}{4}} = \tan^{-1} \frac{1}{\frac{3}{4}} = \cot^{-1} \frac{3}{4}.$
14. $2 \tan^{-1} \frac{1}{2} = \tan^{-1} \frac{2 \cdot \frac{1}{2}}{1 - \frac{1}{4}} = \tan^{-1} \frac{1}{\frac{3}{4}}; \therefore \tan^{-1} \frac{1}{2} = \frac{1}{2} \tan^{-1} \frac{4}{3}.$

$$15. \sin \cot^{-1} x = \sin \operatorname{cosec}^{-1} \sqrt{x^2 + 1} = \frac{1}{\sqrt{x^2 + 1}}, \text{ and}$$

$$\begin{aligned} \cos \tan^{-1} \frac{1}{\sqrt{x^2 + 1}} &= \cos \sec^{-1} \sqrt{\left(\frac{1}{x^2 + 1} + 1\right)} \\ &= \cos \sec^{-1} \sqrt{\left(\frac{x^2 + 2}{x^2 + 1}\right)} = \sqrt{\left(\frac{x^2 + 1}{x^2 + 2}\right)}. \end{aligned}$$

$$16. f = 30^\circ + 45^\circ = l. \quad 17. \text{ Let } \alpha = \tan^{-1} t_1, \quad \beta = \tan^{-1} t_2, \text{ then}$$

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} = \frac{t_1 + t_2}{1 - t_1 t_2}.$$

$$18. f = \tan^{-1} \frac{2 + \sqrt{3} - \frac{1}{\sqrt{3}}}{1 + (2 + \sqrt{3}) \frac{1}{\sqrt{3}}} = \tan^{-1} \frac{2 + 2\sqrt{3}}{2 + 2\sqrt{3}} = \tan^{-1} 1 = l.$$

$$19. f = \tan^{-1} \frac{3}{4} + \tan^{-1} \frac{4}{3} = \tan^{-1} \frac{\frac{3}{4} + \frac{4}{3}}{1 - \frac{3}{4} \cdot \frac{4}{3}} = \tan^{-1} \frac{\frac{25}{12}}{\frac{1}{4}} = \tan^{-1} \frac{25}{3} = l.$$

$$20. f = \tan \tan^{-1} \frac{\frac{1}{2} + \frac{1}{2}}{1 - \frac{1}{2} \cdot \frac{1}{2}} = \tan \tan^{-1} 1 = l.$$

$$21. f = \cot \tan^{-1} \frac{\frac{4}{3} + 7}{1 - \frac{4}{3} \cdot 7} = \cot \tan^{-1}(-1) = \cot \cot^{-1}(-1) = l.$$

$$\begin{aligned} 22. f &= \tan^{-1} \left\{ \left(\frac{2x-y}{y\sqrt{3}} + \frac{2y-x}{x\sqrt{3}} \right) + \left(1 - \frac{(2x-y)(2y-x)}{3xy} \right) \right\} \\ &= \tan^{-1} \left\{ \frac{2(x^2 - xy + y^2)}{xy\sqrt{3}} \times \frac{3xy}{2(x^2 - xy + y^2)} \right\} = \tan^{-1} \sqrt{3} = \frac{1}{2}\pi. \end{aligned}$$

$$\begin{aligned} 23. \sin^{-1} \frac{4}{5} + \sin^{-1} \frac{5}{13} &= \sin^{-1} \left[\frac{4}{5} \sqrt{1 - \frac{25}{169}} + \frac{5}{13} \sqrt{1 - \frac{16}{25}} \right] \\ &= \sin^{-1} \left[\frac{4}{5} \cdot \frac{12}{13} + \frac{5}{13} \cdot \frac{4}{5} \right] = \sin^{-1} \frac{64}{169} = \cos^{-1} \sqrt{1 - \left(\frac{64}{169}\right)^2} = \cos^{-1} \frac{117}{169} \\ &= \frac{1}{2}\pi - \sin^{-1} \frac{117}{169}; \text{ transposing, } f = l. \end{aligned}$$

$$\begin{aligned} 24. f &= \tan^{-1} \frac{\frac{1}{2} + \frac{1}{2}}{1 - \frac{1}{2} \cdot \frac{1}{2}} + \tan^{-1} \frac{\frac{2}{3} + \frac{1}{3}}{1 - \frac{2}{3} \cdot \frac{1}{3}} = \tan^{-1} \frac{4}{3} + \tan^{-1} \frac{3}{2} \\ &= \tan^{-1} \frac{\frac{4}{3} + \frac{3}{2}}{1 - \frac{4}{3} \cdot \frac{3}{2}} = \tan^{-1} 1 = \frac{1}{4}\pi. \end{aligned}$$

$$\begin{aligned} 25. f &= \tan(3 \tan^{-1} x) = \tan \left(\tan^{-1} x + \tan^{-1} \frac{2x}{1-x^2} \right) \\ &= \left(x + \frac{2x}{1-x^2} \right) + \left(1 - \frac{x \cdot 2x}{1-x^2} \right) = \frac{3x-x^3}{1-x^2} \times \frac{1-x^2}{1-3x^2} = l. \end{aligned}$$

$$\begin{aligned} 26. f &= \tan \tan^{-1} \frac{2x}{1-x^2} = \frac{2x}{1-x^2}; \text{ and } l = 2 \tan \tan^{-1} \frac{x+x^3}{1-x^4} = \\ &= 2 \frac{x(1+x^2)}{1-x^4} = \frac{2x}{1-x^2}; \therefore f = l. \end{aligned}$$

$$\begin{aligned} 27. \text{ Let } \alpha &= \sin^{-1} \frac{2\sqrt{ac}}{b}; \quad \cos \alpha = \sqrt{1 - \frac{4ac}{b^2}} = \frac{\sqrt{b^2 - 4ac}}{b}, \\ \therefore f &= -\frac{b}{a} \cdot \frac{1}{2}(1 - \cos \alpha) = -\frac{b}{2a} \left(1 - \frac{\sqrt{b^2 - 4ac}}{b} \right) = l. \end{aligned}$$

28. Let $\alpha = \sin^{-1} \frac{2\sqrt{ac}}{b}$, then $\cos \alpha = \frac{\sqrt{b^2 - 4ac}}{b}$;

$$\therefore -\frac{b}{a} \cdot \frac{1}{2} 2 \cos^2 \frac{1}{2} \alpha = -\frac{b}{2a} (1 + \cos \alpha) = -\frac{b}{2a} \left\{ 1 + \frac{\sqrt{b^2 - 4ac}}{b} \right\} = 7.$$

29. $f = (a+b) \sin \frac{1}{2} C \sqrt{\left\{ 1 + \frac{(a-b)^2 \cos^2 \frac{1}{2} C}{(a+b)^2 \sin^2 \frac{1}{2} C} \right\}}$
 $= \sqrt{[(a+b)^2 \sin^2 \frac{1}{2} C + (a-b)^2 \cos^2 \frac{1}{2} C]} = \sqrt{[(a^2 + b^2)(\cos^2 \frac{1}{2} C + \sin^2 \frac{1}{2} C) - 2ab(\cos^2 \frac{1}{2} C - \sin^2 \frac{1}{2} C)]} = \sqrt{(a^2 + b^2) - 2ab \cos C} = c.$

30. $\sin^{-1}(\tan \frac{1}{4} \pi) = \sin^{-1} 1$ or $\frac{1}{4} \pi$; and $\tan \sin^{-1} \frac{1}{\sqrt{2}} = \tan \frac{1}{4} \pi = 1.$

31. $\tan^{-1} \frac{2x+3x}{1-2x \cdot 3x} = \frac{1}{4} \pi = \tan^{-1} 1$; or $\frac{5x}{1-6x^2} = 1$; or $6x^2 + 5x - 1 = 0$,

$$\therefore x = \frac{-5 \pm \sqrt{25 + 24}}{12} = \frac{1}{3} \text{ or } -1.$$

32. $\tan^{-1} \frac{x - \frac{1}{x}}{1 + x \cdot \frac{1}{x}} = \pi = \tan^{-1} 0$; $x^2 - 1 = 0$, $\therefore x = \pm 1.$

33. $\sin^{-1} \frac{2a}{1+a^2} = \operatorname{cosec}^{-1} \frac{1+a^2}{2a} = \cot^{-1} \sqrt{\left\{ \frac{(1+a^2)^2}{4a} - 1 \right\}}$
 $= \cot^{-1} \frac{1-a^2}{2a} = \tan^{-1} \frac{2a}{1-a^2} = 2 \tan^{-1} a$; $\therefore 2 \tan^{-1} a + 2 \tan^{-1} b = 2 \tan^{-1} x$, $\therefore \tan^{-1} x = \tan^{-1} \frac{a+b}{1-ab}$, $\therefore x = \frac{a+b}{1-ab}.$

34. $\cot^{-1} \frac{1}{x+1} + \cot^{-1} \frac{1}{x-1} = \tan^{-1} \frac{x+1+x-1}{1-(x+1)(x-1)} = \tan^{-1} \frac{2x}{2-x^2}$;

$$\text{and } \tan^{-1} 3x - \cot^{-1} \frac{1}{x} = \tan^{-1} \frac{3x - \frac{1}{x}}{1 + 3x^2} = \tan^{-1} \frac{2x}{1+3x^2}.$$

$$\therefore \frac{2x}{2-x^2} = \frac{2x}{1+3x^2}, \therefore x = 0, \text{ or } 4x^2 = 1, \text{ and } x = \pm \frac{1}{2}.$$

35. $\sec^{-1} a + \sec^{-1} \frac{x}{a} = \cos^{-1} \frac{1}{a} + \cos^{-1} \frac{a}{x}$
 $= \cos^{-1} \left(\frac{a}{ax} - \frac{\sqrt{[(a^2-1)(x^2-a^2)]}}{ax} \right), \sec^{-1} b + \sec^{-1} \frac{x}{b} = \&c.,$

$$\therefore \frac{1}{x} - \frac{\sqrt{[(a^2-1)(x^2-a^2)]}}{ax} = \frac{1}{x} - \frac{\sqrt{[(b^2-1)(x^2-b^2)]}}{bx},$$

$$\therefore \frac{b^2(a^2-1)(x^2-a^2)}{x^2(a^2-b^2)} = \frac{a^2(b^2-1)(x^2-b^2)}{a^2b^2(a^2-b^2)}, \therefore x = \pm ab.$$

36. $\cos^{-1}(-x) - \cos^{-1} x = \cos^{-1}[-x^2 + 1 - x^2] = \sec^{-1} \frac{1}{1-2x^2}$

$$= \tan^{-1} \frac{\sqrt{(4x^2-4x^4)}}{1-2x^2}, \therefore 2\sqrt{(1-x^2)} = \frac{2x\sqrt{(1-x^2)}}{1-2x^2},$$

$$\therefore (1) x^2 - 1 = 0, \text{ and } x = \pm 1; (2) 2x^3 + x - 1 = 0,$$

$$\therefore x = \frac{-1 \pm \sqrt{9}}{4} = \frac{1}{2} \text{ or } -1.$$

CHAPTER XXI.

1. $4 \tan^2 \alpha = 3 \sec^2 \alpha = 3 (\tan^2 \alpha + 1)$; $\therefore \tan^2 \alpha = 3$; $\therefore \tan \alpha = \pm \sqrt{3} = \tan \frac{1}{2}\pi$ or $\tan -\frac{1}{2}\pi$, $\therefore \alpha = n\pi \pm \frac{1}{2}\pi$.
2. $4 \cos^2 A = 3$, $\therefore \cos A = \pm \frac{1}{2}\sqrt{3} = \cos 30^\circ$ or $\cos 150^\circ$,
 $\therefore A = 2n \cdot 180^\circ \pm 30^\circ$ or $2n \cdot 180^\circ \pm 150^\circ = n \cdot 180^\circ \pm 30^\circ$.
3. $\cos A = \sqrt{2} \cos A \sin A$, $\therefore \cos A (\sqrt{2} \sin A - 1) = 0$;
 \therefore (1) $\cos A = 0 = \cos 90^\circ$, $\therefore A = 2n \cdot 180^\circ \pm 90^\circ = n \cdot 180^\circ \pm 90^\circ$;
 (2) $\sin A = \frac{1}{\sqrt{2}} = \sin 45^\circ$, $\therefore A = n \cdot 180^\circ + (-1)^n 45^\circ$.
4. $2 - \cos \theta = \sqrt{3} \sin \theta$, $\therefore 4 - 4 \cos \theta + \cos^2 \theta = 3 \sin^2 \theta = 3(1 - \cos^2 \theta)$,
 $\therefore 4 \cos^2 \theta - 4 \cos \theta + 1 = 0$, or $(2 \cos \theta - 1)^2 = 0$,
 $\therefore \cos \theta = \frac{1}{2} = \cos \frac{1}{3}\pi$, $\therefore \theta = 2n\pi \pm \frac{1}{3}\pi$.
5. $\cos^2 \theta + 4 \cos^2 \theta \sin^2 \theta = 6 \sin^2 \theta$, or $\cos^2 \theta + 4 \cos^2 \theta (1 - \cos^2 \theta) = 6(1 - \cos^2 \theta)$,
 $\therefore 4 \cos^4 \theta - 11 \cos^2 \theta + 6 = 0$, $\therefore \cos^2 \theta = \frac{1}{2} (11 \pm \sqrt{25}) = 2$ (impos.)
 or $\frac{3}{2}$, $\therefore \cos \theta = \pm \frac{1}{2}\sqrt{3}$; \therefore [as in 2] $\theta = n\pi \pm \frac{1}{3}\pi$.
6. $\sec \theta = \sec \pi$, $\therefore \theta = 2n\pi \pm \pi$, i.e. $(2n+1)\pi$.
7. $\cos^2 \alpha = 1$, $\cos \alpha = \pm 1 = \cos 0$ or π , $\alpha = 2n\pi$ or $(2n+1)\pi$, i.e. $n\pi$.
8. $\cot \theta (\cot \theta + \sqrt{3}) = 0$; \therefore (1) $\cot \theta = 0$, $\theta = n\pi + \frac{1}{2}\pi$;
 (2) $\cot \theta = -\sqrt{3} = \cot \frac{2}{3}\pi$, $\theta = n\pi + \frac{2}{3}\pi$.
9. $\sin \theta = \frac{1}{2}(-5 \pm \sqrt{9}) = -\frac{1}{2}$ or -2 (impos.); $\therefore \theta = n\pi + (-1)^n \frac{3}{2}\pi$.
10. $\sin \theta (2 \cos \theta + 1) = 0$; \therefore (1) $\sin \theta = 0 = \sin 0$, $\therefore \theta = n\pi$;
 (2) $\cos \theta = -\frac{1}{2} = \cos \frac{2}{3}\pi$, $\therefore \theta = 2n\pi \pm \frac{2}{3}\pi$.
11. $2 \sin 2\theta \cos \theta + \sin 2\theta = 0$, or $\sin 2\theta (2 \cos \theta + 1) = 0$;
 \therefore (1) $\sin 2\theta = 0$, $\therefore 2\theta = n\pi$. (2) As (2) in 10.
12. $2 \cos 2\theta \cos \theta + \cos 2\theta = 0$, or $\cos 2\theta (2 \cos \theta + 1) = 0$;
 \therefore (1) $\cos 2\theta = 0$, $\therefore 2\theta = 2n\pi \pm \frac{1}{2}\pi = n\pi + \frac{1}{2}\pi$. (2) As (2) in 10.
13. $2 \cos \frac{3}{2}\theta \cos \frac{1}{2}\pi - \cos \frac{3}{2}\theta = 0$, or $\cos \frac{3}{2}\theta (2 \cos \frac{1}{2}\theta - 1) = 0$;
 (1) $\cos \frac{3}{2}\theta = 0$, $\therefore \frac{3}{2}\theta = n\pi + \frac{1}{2}\pi$ [as (1) in 3];
 (2) $\cos \frac{1}{2}\theta = \frac{1}{2}$, $\therefore \frac{1}{2}\theta = 2n\pi \pm \frac{1}{3}\pi$.
14. $2 \cos 4\theta \sin 3\theta - \sin 3\theta = 0$, $\therefore \sin 3\theta (2 \cos 4\theta - 1) = 0$;
 \therefore (1) $\sin 3\theta = 0$, $3\theta = n\pi$; (2) $\cos 4\theta = \frac{1}{2}$, $4\theta = 2n\pi \pm \frac{1}{2}\pi$.
15. $\cos \theta \frac{1}{\sqrt{2}} + \sin \theta \frac{1}{\sqrt{2}} = \frac{1}{2}$, or $\cos(\theta - \frac{1}{2}\pi) = \frac{1}{2}$, $\theta - \frac{1}{2}\pi = 2n\pi \pm \frac{1}{3}\pi$.
16. $\sin \theta \frac{1}{\sqrt{2}} + \cos \theta \frac{1}{\sqrt{2}} = 1$, or $\sin(\theta + \frac{1}{2}\pi) = 1$, $\theta + \frac{1}{2}\pi = 2n\pi + \frac{1}{2}\pi$.
17. $\sin \theta \frac{\sqrt{3}}{2} - \cos \theta \frac{1}{2} = \frac{1}{\sqrt{2}}$, or $\sin(\theta - \frac{1}{6}\pi) = \frac{1}{\sqrt{2}}$, $\theta - \frac{1}{6}\pi = n\pi + (-1)^n \frac{1}{4}\pi$.
18. $\cos \theta \frac{1}{\sqrt{2}} - \sin \theta \frac{1}{\sqrt{2}} = 1$, or $\cos(\theta + \frac{1}{2}\pi) = 1$, $\theta + \frac{1}{2}\pi = 2n\pi$.
19. $\tan(\frac{1}{2}\pi + \alpha) \tan(\frac{1}{2}\pi + \alpha) = 3$, $\therefore \tan(\frac{1}{2}\pi + \alpha) = \pm \sqrt{3}$, $\frac{1}{2}\pi + \alpha = n\pi \pm \frac{1}{3}\pi$.
20. $2 \cos(n-1)\theta \cos \theta = \cos \theta$, or $\cos \theta [2 \cos(n-1)\theta - 1] = 0$;

- $\therefore (1) \cos \theta = 0, \theta = n\pi + \frac{1}{2}\pi; (2) \cos (n-1)\theta = \frac{1}{2}, (n-1)\theta = 2n\pi \pm \frac{1}{2}\pi.$
21. $\sec \theta = \frac{1}{2} [-2\sqrt{2}-1 \pm \sqrt{(9+4\sqrt{2}-8\sqrt{2})}] = -\sqrt{2} \text{ or } -\frac{1}{2} (\text{impos.}); \theta = 2n\pi \pm \frac{1}{2}\pi.$
22. $\cos 4a + \cos 2a = \cos 12a + \cos 2a, \therefore 2 \sin 8a \sin 4a = 0;$
 $\therefore (1) \sin 8a = 0, \therefore 8a = n\pi; (2) \sin 4a = 0, 4a = n\pi.$
23. $\cos a = \sin a - 4 \sin a (1 - \sin^2 a) = 4 \sin^3 a - 3 \sin a = -\sin 3a;$
 $\therefore \cos (\frac{1}{2}\pi + 3a) = \cos a, \therefore \frac{1}{2}\pi + 3a = 2n\pi \pm a.$
24. $\sin a - \cos a = 2\sqrt{2} \sin a \cos a = \sqrt{2} \sin 2a, \therefore 1 - \sin 2a = 2 \sin^2 2a,$
 $\therefore 2 \sin^2 2a + \sin 2a - 1 = 0, \therefore \sin 2a = \frac{1}{2} (-1 \pm \sqrt{9}) = \frac{1}{2} \text{ or } -1,$
 $\therefore 2a = n\pi + (-1)^n \frac{1}{2}\pi \text{ or } 2n\pi - \frac{1}{2}\pi.$

CHAPTER XXII.

1. Since 300° is in 4th quad., in which the tangent is negative,
 $\tan 300^\circ = -\sqrt{\sec^2 300^\circ - 1} = -\sqrt{4-1} = -\sqrt{3}.$
2. Since $\tan A = \tan (n \cdot 180^\circ + A), \sqrt{\tan^2 A + 1}$ must give, for all values of n , $\sec (n \cdot 180^\circ + A).$
Geom.—But $n \cdot 180^\circ + A$, when n is even, gives one group of angles; when n is odd, another; and (fig., p. 126) if $\angle OAP = A$, the one group is bounded by OA and OP , the other by OA and OP' . The two groups are thus in vertically opposite quadrants; and their secants therefore differ in sign.
Analyt.—(1) If n be even, and $= 2m$ suppose, then
 $\sec (n \cdot 180^\circ + A) = \sec (m \cdot 360^\circ + A) = \sec A.$
 (2) If n be odd, and $= 2m + 1$ suppose, then
 $\sec (n \cdot 180^\circ + A) = \sec (m \cdot 360^\circ + 180^\circ + A) = \sec (180^\circ + A) = -\sec A.$
3. $f = \frac{1}{2}\sqrt{2+2\cos\frac{1}{2}\pi} = \frac{1}{2}\sqrt{2+2} = \frac{1}{2}\sqrt{2+2\cos\frac{1}{2}\pi} = l.$
4. $f = \frac{1}{2}\sqrt{2-2\cos\frac{1}{8}\pi} = \frac{1}{2}\sqrt{2-2\cdot\frac{1}{2}\sqrt{2+\sqrt{2}+\sqrt{2}}} = l.$
5. $f = \frac{1}{2}\sqrt{2+2\cos\frac{1}{4}\pi} = \frac{1}{2}\sqrt{2+2\cdot\frac{1}{2}\sqrt{2+2\cos\frac{1}{2}\pi}} = l.$
6. $f = \frac{1}{2}\sqrt{2-2\cos\frac{1}{4}\pi} = \frac{1}{2}\sqrt{2-2\cdot\frac{1}{2}\sqrt{2+\sqrt{2}+\sqrt{3}}} = l.$
7. $f = \frac{1}{2}\sqrt{2+2\cos\frac{1}{10}\pi} = \frac{1}{2}\sqrt{2+2\cdot\frac{1}{2\sqrt{2}}\sqrt{5+\sqrt{5}}} = l.$
8. $f = \frac{1}{2}\sqrt{2-2\cos\frac{1}{8}\pi} = \frac{1}{2}\sqrt{2-2\cdot\frac{1}{2}\sqrt{2+\sqrt{\frac{1}{2}}(5+\sqrt{5})}} = l.$
9. Divide a revolution into eight semi-quadrants. Now the limits $n\pi + \frac{1}{4}\pi$ and $n\pi + \frac{3}{4}\pi$, when n is even, embrace the 2nd and 3rd semi-quads.; when n is odd, the 6th and 7th. Here (fig., p. 41) $PN > ON$, and \therefore sine $>$ cosine. $n\pi - \frac{1}{4}\pi$ and $n\pi + \frac{1}{4}\pi$, when n is even, embrace the 8th and 1st; when n is odd, the 4th and 5th. Here $ON > PN$, and \therefore cosine $>$ sine.
10. $\sin A - \cos A = \sqrt{2} \sin (A - 45^\circ)$; and, in sign, is +ve, when $A - 45^\circ$ is between 0 and 180° , and \therefore when A is between 45° and 225° ; -ve, when $A - 45^\circ$ is between 180° and 360° , and \therefore for A from 225° to 360° , and 0 to 45° .

In magnitude, as A increases positively, it = 0 when $A = 45^\circ$, increases to 1 when $A = 135^\circ$, decreases to 0 when $A = 225^\circ$, increases to -1 when $A = 315^\circ$, decreases to 0 at 405° .

11. $\tan A + \cot A = \frac{\sin^2 A + \cos^2 A}{\sin A \cos A} = \frac{2}{\sin 2A} = 2 \operatorname{cosec} 2A$; and, in sign, is +ve, when $2A$ is between 0 and 180° , and \therefore when A is between 0 and 90° ; -ve, when $2A$ is between 180° and 360° , and \therefore when A is between 90° and 180° .

In magnitude, as A increases positively, it = ∞ when $A = 0$, decreases to 1 when $A = 45^\circ$, increases to ∞ when $A = 90^\circ$, decreases to -1 when $A = 135^\circ$, increases to ∞ when $A = 180^\circ$.

The variation when A is in 3rd and 4th quads. repeats the above.

12. $\cos 15^\circ > \sin 15^\circ$ and +ve, $\therefore \sin 15^\circ + \cos 15^\circ = \sqrt{1 + \sin 30^\circ} = \sqrt{1 + \frac{1}{2}} = \sqrt{\frac{3}{2}}$; and $\sin 15^\circ - \cos 15^\circ = -\sqrt{1 - \sin 30^\circ} = -\sqrt{\frac{1}{2}}$. Adding, $2 \sin 15^\circ = \frac{\sqrt{3}-1}{\sqrt{2}}$, $\therefore \sin 15^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}$.
13. $\sin 75^\circ > \cos 75^\circ$ and +ve, $\therefore \sin 75^\circ + \cos 75^\circ = \sqrt{1 + \sin 150^\circ} = \sqrt{\frac{3}{2}}$, and $\sin 75^\circ - \cos 75^\circ = \sqrt{1 - \sin 30^\circ} = \sqrt{\frac{1}{2}}$. Adding, $2 \sin 75^\circ = \frac{\sqrt{3}+1}{2}$, $\therefore \sin 75^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}$.
14. $\sin (-105^\circ) > \cos (-105^\circ)$ and -ve, $\therefore \sin (-105^\circ) + \cos (-105^\circ) = -\sqrt{1 + \sin (-210^\circ)} = -\sqrt{1 + \frac{1}{2}} = -\frac{\sqrt{3}}{\sqrt{2}}$, and $\sin (-105^\circ) - \cos (-105^\circ) = -\sqrt{1 - \sin (-210^\circ)} = -\frac{1}{\sqrt{2}}$. Add and divide, $\sin (-105^\circ) = \frac{-\sqrt{3}-1}{2\sqrt{2}}$.
15. $\cos (-165^\circ) > \sin (-165^\circ)$ and -ve, $\therefore \sin (-165^\circ) + \cos (-165^\circ) = -\sqrt{1 + \sin (-330^\circ)} = -\sqrt{\frac{3}{2}}$, and $\sin (-165^\circ) - \cos (-165^\circ) = -\sqrt{1 - \sin (-330^\circ)} = \sqrt{\frac{1}{2}}$. Add and divide, $\sin (-165^\circ) = \frac{-\sqrt{3}+1}{2\sqrt{2}}$.
16. $\cos (-15^\circ) > \sin (-15^\circ)$ and +ve, $\therefore \sin (-15^\circ) + \cos (-15^\circ) = \sqrt{1 + \sin (-30^\circ)} = \sqrt{1 - \frac{1}{2}} = \sqrt{\frac{1}{2}}$, and $\sin (-15^\circ) - \cos (-15^\circ) = -\sqrt{1 - \sin (-30^\circ)} = -\sqrt{1 + \frac{1}{2}} = -\sqrt{\frac{3}{2}}$. Subtract and divide, $\cos (-15^\circ) = \frac{1+\sqrt{3}}{2\sqrt{2}}$.
17. $\sin (-75^\circ) > \cos (-75^\circ)$ and -ve, $\therefore \sin (-75^\circ) + \cos (-75^\circ) = -\sqrt{1 + \sin (-150^\circ)} = -\sqrt{1 - \frac{1}{2}} = -\sqrt{\frac{1}{2}}$, and $\sin (-75^\circ) - \cos (-75^\circ) = -\sqrt{1 - \sin (-150^\circ)} = -\sqrt{1 + \frac{1}{2}} = -\sqrt{\frac{3}{2}}$. Subt. and div., $\cos (-75^\circ) = \frac{-1+\sqrt{3}}{2\sqrt{2}}$.
18. $\sin 105^\circ > \cos 105^\circ$ and +ve, $\therefore \sin 105^\circ + \cos 105^\circ = \sqrt{1 + \sin 210^\circ} = \sqrt{1 - \frac{1}{2}} = \frac{1}{\sqrt{2}}$, and $\sin 105^\circ - \cos 105^\circ = \sqrt{1 - \sin 210^\circ} = \sqrt{1 + \frac{1}{2}} = \sqrt{\frac{3}{2}}$. Subt. and div., $\cos 105^\circ = \frac{1-\sqrt{3}}{2\sqrt{2}}$.

19. $\cos 165^\circ > \sin 165^\circ$ and $-ve$, $\therefore \sin 165^\circ + \cos 165^\circ =$
 $-\sqrt{1 + \sin 330^\circ} = -\sqrt{1 - \frac{1}{2}} = \frac{-1}{\sqrt{2}}$, and $\sin 165^\circ - \cos 165^\circ$
 $= \sqrt{1 - \sin 330^\circ} = \sqrt{1 + \frac{1}{2}} = \sqrt{\frac{3}{2}}$. Subt. and div., $\cos 165^\circ =$
 $\frac{-1 - \sqrt{3}}{2\sqrt{2}}$.
20. $\cos 9^\circ > \sin 9^\circ$ and $+ve$, $\therefore \sin 9^\circ + \cos 9^\circ = \sqrt{1 + \sin 18^\circ} =$
 $\sqrt{1 + \frac{1}{2}(\sqrt{5} - 1)} = \frac{1}{2}\sqrt{3 + \sqrt{5}}$, and $\sin 9^\circ - \cos 9^\circ = -\sqrt{1 - \sin 18^\circ}$
 $= -\sqrt{1 - \frac{1}{2}(\sqrt{5} - 1)} = -\frac{1}{2}\sqrt{5 - \sqrt{5}}$. Add and divide,
 $\sin 9^\circ = \frac{1}{2}[\sqrt{3 + \sqrt{5}} - \sqrt{5 - \sqrt{5}}]$.
21. $\frac{1}{2}A$ is between 90° and 135° , $\therefore \sin \frac{1}{2}A > \cos \frac{1}{2}A$ and $+ve$,
 $\therefore \sin \frac{1}{2}A + \cos \frac{1}{2}A = \sqrt{1 + \sin A} = \sqrt{\frac{5}{2}}$, and $\sin \frac{1}{2}A - \cos \frac{1}{2}A =$
 $\sqrt{1 - \sin A} = \sqrt{\frac{3}{2}}$. Subtract and divide, $\cos \frac{1}{2}A = \frac{\sqrt{6} - 2}{2\sqrt{5}}$
 $= \frac{\sqrt{3} - \sqrt{2}}{\sqrt{10}}$.
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EXAMINATION PAPERS.

I.

1. $25^\circ - 25^\circ = 25^\circ - \frac{1}{10} 25^\circ = \frac{9}{10} 25^\circ = 2\frac{1}{2}^\circ$. $25^\circ - 25^\circ = \frac{1}{10} 25^\circ - 25^\circ =$
 $\frac{1}{10} 25^\circ = 2\frac{1}{2}^\circ$.
2. Since a right angle in the respective measures $= 90^\circ = 100^\circ = \frac{1}{2}\pi$,
 $\therefore G : D = 100 : 90 = 10 : 9$, and $G : 20C = 100 : 20\frac{1}{2}\pi = 10 : \pi$;
 $\therefore \frac{G}{10} = \frac{D}{9}$, and also $= \frac{20C}{\pi}$.
3. $3(\sin^2 \frac{1}{2}A + \cos^2 \frac{1}{2}A) = 3 \times 1 = 3$.
4. $2 - v = 1 + 1 - v = 1 + \cos A$, $\therefore \sin^2 A = 1 - \cos^2 A =$
 $(1 - \cos A)(1 + \cos A) = v(2 - v)$, $\therefore f = l$.
5. In fig., p. 17, let $A = B = 45^\circ$ (Euc. I. 32); then $AC = BC$
 (Euc. I. 6); $\therefore \frac{BC}{AC} = 1$, or $\tan A = \cot B = 1$.
6. covers $(90^\circ - A) = 1 - \sin(90^\circ - A) = 1 - \cos A = \text{vers } A$.

II.

1. vertical angle $= 2(90^\circ - 87^\circ 15' 13'' \cdot 95) = 2 \times 2^\circ 44' 46'' \cdot 05$
 $= 5^\circ 29' 32'' \cdot 1 = 5^\circ 49' 22\frac{1}{2}'' = \frac{1}{10} \text{ of } 5^\circ \cdot 49225 = 6^\circ \cdot 1025 = 6^\circ 10' 25''$.
2. Circ. meas. of angle $= \frac{30}{25 \times 12} = \frac{1}{10}$; then $\frac{1}{10} \pi : \frac{1}{10} :: 90^\circ : \text{required}$
 sexag. meas. $= \frac{90^\circ \times 2}{3 \cdot 1416 \times 10} = \frac{180^\circ}{31 \cdot 416} = 5 \frac{23 \frac{1}{2}}{100}^\circ$.

3. $f = 1 \times 1$, and $l = 1$; $\therefore f = l$.
4. Constructing as on p. 22, $BA = BC = 2BD$, $\therefore \frac{BA}{BD} = 2$,
 $\therefore \sec 60^\circ = 2$.
5. $f = \left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right) + \left(\frac{1}{2} - \frac{\sqrt{3}}{2}\right) = \frac{1 + \sqrt{3}}{2} \times \frac{2}{1 - \sqrt{3}} = \frac{1 + \sqrt{3}}{1 - \sqrt{3}} = l$.
6. In fig. on p. 22, let $BA = AC = a$, and AD be p ; then $2AD = BA \sqrt{3}$, $\therefore \frac{AD}{BA} = \frac{\sqrt{3}}{2}$, or $\sin B = \frac{\sqrt{3}}{2} = \sin 60^\circ$, $\therefore B = C = 60^\circ$.

III.

1. Angle $= -360^\circ - \frac{2}{3} \cdot 360^\circ = -360^\circ - 312^\circ = -672^\circ$. For circ. meas.,
 $90^\circ : -672^\circ = \frac{\pi}{2} : -\frac{\pi \times 672}{2 \times 90} = -\frac{56}{15} \pi = -\frac{3 \cdot 1416 \times 56}{15} = -11 \cdot 72864$.
2. $l = \text{vers } A (1 - \cos A) = (1 - \cos A)(1 + \cos A) - 1 - \cos^2 A = f$.
3. $\cot A = \frac{a}{x}$, $\therefore \text{cosec } A = \sqrt{1 + \frac{a^2}{x^2}} = \frac{\sqrt{(x^2 + a^2)}}{x}$, $\therefore \sin A = \frac{x}{\sqrt{(a^2 + x^2)}}$.
4. $f = \sin^2 a (1 - \cos^2 \beta) + \cos^2 a (1 - \sin^2 \beta) + \sin^2 a \cos^2 \beta + \cos^2 a \sin^2 \beta$
 $= \sin^2 a + \cos^2 a = 1$.
5. $\tan(A + B) = \sqrt{3} = \tan 60^\circ$, $\therefore A + B = 60^\circ$; similarly, $A - B = 30^\circ$.
 Adding, $2A = 90^\circ$; subtracting, $2B = 30^\circ$; $\therefore A = 45^\circ$, $B = 15^\circ$.
6. Let A, B (fig. 30) be the two objects, D, C the successive points of observation; reqd. AB . Given $CD = 200$ yds., $\angle ADC = 67^\circ 30'$, $\angle ADB = 7^\circ 30'$; $\therefore \angle BDC = 60^\circ$. Now $AC = CD \tan \angle ADC = CD \times 2 \cdot 414$, and $BC = CD \tan 60^\circ = CD \times 1 \cdot 732$, $\therefore AB = CD (2 \cdot 414 - 1 \cdot 732) = 200 \text{ yds.} \times \cdot 682 = 136 \cdot 4 \text{ yds.}$

IV.

1. A straight line is the shortest distance between two points; therefore, if A, B be joined in fig. of Art. 7, arc $AB >$ chord AB . Now let $O = 60^\circ$; then (Euc. I. 32) the triangle ABC would be equiangular, and \therefore chord $AB = AO =$ radius; \therefore arc $AB >$ radius; \therefore angle $O >$ unit of circ. meas.; or $60^\circ >$ unit of circ. meas.
2. $\sin^2 \theta = p^2 - 2pq + q^2$, $4pq = \cos^2 \theta$;
 $\therefore 1 = \sin^2 \theta + \cos^2 \theta = p^2 + 2pq + q^2 = (p + q)^2$; $\therefore f = l$.
3. $f = \frac{\sin^2 A \sin^2 B - \cos^2 A \cos^2 B}{\cos^2 A \cos^2 B} = \frac{\sin^2 A (1 - \cos^2 B) - (1 - \sin^2 A) \cos^2 B}{\cos^2 A \cos^2 B} = l$.
4. Let A, B be the acute angles, and $B > A$. Then complement of $B = A$, comp. of $90^\circ = 0$; $\therefore 0, A$, and $90^\circ - A$ are in A. P.;
 $\therefore 2A = 0 + 90^\circ - A$; $\therefore A = 30^\circ$, $\therefore B = 60^\circ$.
5. Divide 1st by 2nd eq.; then $\cos A = \frac{\sqrt{2}}{\sqrt{3}} \cos B$; $\therefore \cos^2 A = \frac{2}{3} \cos^2 B$,

and $\sin^2 A = 2 \sin^2 B$. Adding, $1 = \frac{4}{3} \cos^2 B + 2 \sin^2 B = -\frac{4}{3} \cos^2 B + 2$,
 $\therefore \cos B = \frac{\sqrt{3}}{2}$, and $\cos A = \frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{2} = \frac{1}{\sqrt{2}}$; $\therefore B = 30^\circ$, $A = 45^\circ$.

6. In fig. (8), let BC , DA be the towers. Given $FBD (= BDE) = 30^\circ$,
 $DE = 30$ yds., $BC = 100$ yds. Now $BE = DE \tan 30^\circ = 30$ yds. $\times \frac{1}{\sqrt{3}}$
 $= 10\sqrt{3}$ yds. $= 17.32$ yds.; $\therefore AD = CE = 100$ yds. $- 17.32$ yds. $=$
 82.68 yds.

V.

1. Radius of equator $= 4000$ m.; \therefore required angle in circ. meas. $=$
 $\frac{1309}{4000} = \frac{31416}{4000 \times 24} = \frac{5}{48} \pi$, and in degrees, &c. $= \frac{1309}{4000} \cdot \frac{180^\circ}{\pi} =$
 $\frac{1309}{4000} \cdot \frac{180^\circ}{3.1416} = \frac{1800^\circ}{4 \times 24} = 18^\circ 45'$.

$$2. f = \frac{1 - \tan^2 A \tan^2 B}{\tan^2 A \tan^2 B} \cdot \frac{\cos^2 A \cos^2 B}{\cos^2 A \cos^2 B} = \frac{\cos^2 A \cos^2 B - \sin^2 A \sin^2 B}{\sin^2 A \sin^2 B}$$

$$= \frac{\cos^2 A (1 - \sin^2 B) - (1 - \cos^2 A) \sin^2 B}{\sin^2 A \sin^2 B} = l.$$

3. Squaring (1), $\sin^2 \alpha + 2 \sin \alpha \sin \beta + \sin^2 \beta = 2$; from (2), $4 \sin \alpha \sin \beta = 2$; subtracting, $(\sin \alpha - \sin \beta)^2 = 0$, $\therefore \sin \alpha = \sin \beta$; and from (2), $\sin^2 \alpha = \frac{1}{2}$, $\therefore \sin \alpha = \frac{1}{\sqrt{2}}$, and $\alpha = \frac{1}{2}\pi = \beta$.

4. $\tan^2 \theta + 1 = 2 \tan \theta$, $\therefore (\tan \theta - 1)^2 = 0$, $\therefore \tan \theta = 1$, $\therefore \theta = \frac{1}{2}\pi$;
 $\therefore \sin \theta + \cos \theta = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$.

5. Remembering that $\frac{1}{2}\theta$ varies from 0 to π , while θ varies from 0 to 2π ; we have simply to trace $2 \operatorname{cosec} \frac{1}{2}\theta$ through the first two quadrants, in which it is +ve. Now, in fig. of Art. 45, $\operatorname{cosec} A = \frac{r}{y}$, and r does not vary.

When $\frac{1}{2}\theta$ is	0	between 0 and $\frac{1}{2}\pi$	$\frac{1}{2}\pi$	between $\frac{1}{2}\pi$ and π	π
y is	0	increasing	r	decreasing	0
$\therefore 2 \operatorname{cosec} \frac{1}{2}\theta$ is	∞	decreasing	2	increasing	∞

6. In fig. (31), take C , D , H as Calais, Dover, and Hastings; then
 $CD = 13\sqrt{3}$ m., $DH = 39$ m., $\angle CDH = 56\frac{1}{4}^\circ + 33\frac{1}{4}^\circ = 90^\circ$,
 $\therefore \tan H = \frac{CD}{DH} = \frac{13\sqrt{3}}{39} = \frac{1}{\sqrt{3}}$, $\therefore H = 30^\circ$,
 $\therefore CHE = 56\frac{1}{4}^\circ - 30^\circ = 26\frac{1}{4}^\circ$.

VI.

1. Let $(n-x)^\circ$ and x° be required parts; then $60(n-x) = \frac{1}{2}x \times 100$;
 $\therefore 64(n-x) = 100x$, $\therefore x = \frac{64}{164}n = \frac{16}{41}n$, and $n-x = \frac{25}{41}n$.
 2. $\therefore \sec^2 \theta - \tan^2 \theta = 1$, $\therefore (\sec^2 \theta - \tan^2 \theta)^3 = 1$; squaring, &c., $f = l$.

3. $\frac{\cos^2 A}{\sin^2 C} + \frac{\cos^2 B}{\sin^2 C} = \cos^2 \theta + \cos^2 (90^\circ - \theta) = \cos^2 \theta + \sin^2 \theta = 1$,
 $\therefore 1 - \sin^2 A + 1 - \sin^2 B = \sin^2 C$, $\therefore f = l$.
4. In fig. to Art. 45, $\tan A = \frac{PN}{ON}$, $\sin A = \frac{PN}{OP}$. Now, as A varies, ON is always less than OP in the right-angled triangle PON , unless they coincide, which happens when $A = n. 180^\circ$, $\therefore \tan A > \sin A$, unless $A = n. 180^\circ$.
- In same fig., $\sec A = \frac{OP}{ON}$, $\tan A = \frac{PN}{ON}$, and OP is always $> PN$, except when they coincide, which happens when $A = n. 180^\circ + 90^\circ$,
 $\therefore \sec A > \tan A$, except $A = n. 180^\circ + 90^\circ$.
5. $\sin 150^\circ = \sin (180^\circ - 150^\circ) = \sin 30^\circ = \frac{1}{2}$.
 $\sec 585^\circ = \sec (585^\circ - 360^\circ) = \sec 225^\circ = \sec (180^\circ + 45^\circ)$
 $= -\sec 45^\circ = -\sqrt{2}$.
 $\cot \frac{3}{8}\pi = \cot (4\pi - \frac{1}{8}\pi) = \cot (-\frac{1}{8}\pi) = -\cot \frac{1}{8}\pi = -\sqrt{3}$.
6. Taking fig. (32), OCA is a right-angled triangle (Euc. III. 18); and
 $\therefore BAC = 60^\circ$ and $BA = BC$, $\therefore ABC$ is equilateral;
 $\therefore OC = CA \tan 30^\circ = t \frac{1}{\sqrt{3}}$, and $BC = CA = t$.

VII.

1. $f = \frac{\tan A - \sin A \cos A}{\tan A \cos A} = \frac{\sin A - \sin A \cos A}{\sin A} = 1 - \cos A$, and
 $l = \frac{\operatorname{cosec} A - \cot A \sin A}{\operatorname{cosec} A \sin A} = 1 - \cos A$; $\therefore f = l$.
2. $f = \frac{\sin^3 \theta}{\cos \theta} + \frac{\cos^3 \theta}{\sin \theta} + 2 \sin \theta \cos \theta = \frac{\sin^4 \theta + 2 \sin^2 \theta \cos^2 \theta + \cos^4 \theta}{\cos \theta \sin \theta}$
 $= \frac{(\sin^2 \theta + \cos^2 \theta)^2}{\cos \theta \sin \theta} = \frac{\sin^2 \theta + \cos^2 \theta}{\cos \theta \sin \theta} = \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} = l$.
3. By hypothesis, $\frac{1}{n^2 \sin^2 A} - \frac{\tan^2 B}{\sin^2 A} = \sec^2 \phi - \tan^2 \phi = 1$.
 $\therefore \frac{\tan^2 B}{\sin^2 A} = \frac{1}{n^2 \sin^2 A} - 1$; $\therefore n^2 \tan^2 B = 1 - n^2 \sin^2 A$, or
 $n^2 (\sec^2 B - 1) = 1 - n^2 (1 - \cos^2 A)$, $\therefore n^2 \sec^2 B = 1 + n^2 \cos^2 A$,
 $\therefore \cos^2 B = \&c.$
4. $5 = \sin 30^\circ = \sin 150^\circ$; adding +ve and -ve revolutions, by Art. 58, we get from 150° , -210° , -570° , 510° ; and from 30° , -330° , -690° , 390° , 750° .
5. Let (fig. 33) $AOB = 2A$, AB being an arc of centre O ; draw BC perp. to OA , and OD bisecting AOB . Then, $AB > BC$, $\therefore ACB = 90^\circ$ and is therefore $> BAC$, $\therefore 2AD > BC$, $\therefore 2 \frac{AD}{OA} > \frac{BC}{OA}$, i.e. $> \frac{BC}{OB}$.
6. $\tan (90^\circ + A) = -\cot A$ (Art. 60), $\therefore f = 1 - \tan A \cot A = 1 - 1 = 0$.

$$7. f = (\operatorname{cosec}^2 A - 1) - (\sec^2 A - 1) = \frac{1}{\sin^2 A} - \frac{1}{\cos^2 A} = \frac{\cos^2 A - \sin^2 A}{\sin^2 A \cos^2 A} \\ = \frac{4 \cos 2A}{\sin^2 2A} = l.$$

$$8. f = \cos A \cos 2A + \sin A \sin 2A - (\sin A \cos 2A + \cos A \sin 2A) \\ = \cos (2A - A) - \sin (A + 2A) = l.$$

VIII.

$$1. 1309^\circ + 2^\circ 24' = \frac{1}{4}(3 \cdot 1416^\circ) + 2\frac{2}{3}^\circ = \frac{1}{4}\pi^\circ + 1\frac{2}{3}^\circ = \frac{200^\circ}{3} + 1\frac{2}{3}^\circ \times \frac{1}{6} \\ = \frac{25^\circ}{3} + \frac{2^\circ}{3} = \frac{27^\circ}{3} = 9^\circ.$$

$$2. f = \frac{\cos \theta}{\sin \theta} - \frac{\cos^2 \theta}{\sin \theta} + \frac{\sin \theta}{\cos \theta} = \frac{\cos \theta}{\sin \theta} + \frac{\sin \theta}{\cos \theta} - \frac{1 - \sin^2 \theta}{\sin \theta} \\ = \frac{\cos^2 \theta + \sin^2 \theta}{\sin \theta \cos \theta} - \frac{1}{\sin \theta} + \sin \theta = \operatorname{cosec} \theta \sec \theta - \operatorname{cosec} \theta + \sin \theta = l.$$

$$3. 1 = \operatorname{cosec}^2 \theta - \cot^2 \theta = \frac{\sin^2 B}{\cos^2 A} - \frac{n^2}{m^2 \cos^2 A}; \therefore m^2 \cos^2 A = m^2 \sin^2 B - n^2, \\ \therefore m^2 \sin^2 B = m^2 \cos^2 A + n^2, \therefore f = l.$$

$$4. \operatorname{cosec} \theta = \frac{2 + \sqrt{3} \pm \sqrt{(7 + 4\sqrt{3} - 8\sqrt{3})}}{2\sqrt{3}} = \frac{2 + \sqrt{3} \pm (2 - \sqrt{3})}{2\sqrt{3}} = \frac{4}{2\sqrt{3}} \\ \text{or } \frac{2\sqrt{3}}{2\sqrt{3}} = \frac{2}{\sqrt{3}} \text{ or } 1; \therefore \theta = \frac{1}{3}\pi \text{ or } \frac{2}{3}\pi \text{ or } \frac{4}{3}\pi.$$

$$5. \frac{2n\sqrt{(1-n^2)}}{1-2n^2} = \frac{\sin 2A}{\cos 2A} = \frac{2 \sin A \cos A}{1-2 \sin^2 A} = \frac{2 \sin A \sqrt{(1-\sin^2 A)}}{1-2 \sin^2 A}; \\ \therefore \sin A = n, \text{ by inspection.}$$

$$6. l = \frac{\sin(\frac{1}{2}\pi + \theta) \cos(\frac{1}{2}\pi - \theta) - \cos(\frac{1}{2}\pi + \theta) \sin(\frac{1}{2}\pi - \theta)}{\sin(\frac{1}{2}\pi + \theta) \cos(\frac{1}{2}\pi - \theta) + \cos(\frac{1}{2}\pi + \theta) \sin(\frac{1}{2}\pi - \theta)} \\ = \frac{\sin(\frac{1}{2}\pi + \theta - \frac{1}{2}\pi + \theta)}{\sin(\frac{1}{2}\pi + \theta + \frac{1}{2}\pi - \theta)} = \frac{\sin 2\theta}{\sin \frac{1}{2}\pi} = f.$$

$$7. 2f = \sin 2A - \sin 2B + \sin 2B - \sin 2C + \sin 2C - \sin 2A = 0.$$

$$8. f = 2 \sin(A+B) \cos 2B \sec 2B = 2 \sin(A+B) = \\ 2 \sin(A+B) \sin(A-B) \operatorname{cosec}(A-B) = (\cos 2B - \cos 2A) \operatorname{cosec}(A-B).$$

IX.

$$1. \cos^2 A + 2 \cos A \cos B + \cos^2 B = \frac{1}{4}(4 + 2\sqrt{3}), \text{ and } 4 \cos A \cos B = \\ \frac{1}{4}(4\sqrt{3}); \therefore \cos^2 A - 2 \cos A \cos B + \cos^2 B = \frac{1}{4}(4 - 2\sqrt{3}); \\ \therefore \cos A - \cos B = \frac{1}{2}(1 - \sqrt{3}). \text{ Adding, \&c., } \cos A + \cos B = \frac{1}{2}(1 + \sqrt{3}), \\ 2 \cos A = 1, 2 \cos B = \sqrt{3}, \therefore A = 60^\circ, B = 30^\circ.$$

$$2. f = \sin \theta + \frac{\sin^2 \theta}{\cos \theta} + \cos \theta + \frac{\cos^2 \theta}{\sin \theta} - \sec \theta \\ = \sin \theta + \frac{\cos^2 \theta}{\sin \theta} + \frac{\sin^2 \theta}{\cos \theta} + \cos \theta - \sec \theta \\ = \frac{\sin^2 \theta + \cos^2 \theta}{\sin \theta} + \frac{\sin^2 \theta + \cos^2 \theta}{\cos \theta} - \sec \theta = \operatorname{cosec} \theta + \sec \theta - \sec \theta = \operatorname{cosec} \theta \\ = \sqrt{(\cot^2 \theta + 1)} = \sqrt{\left(\frac{a^2 - b^2}{b^2} + 1\right)} = \sqrt{\frac{a^2}{b^2}} = l.$$

4. $L \cot 35^\circ 20' = 10 + \log \frac{1}{\tan 35^\circ 20'} = 10 - \log \tan 35^\circ 20'$
 $= 10 - (L \tan 35^\circ 20' - 10) = 20 - 9.850593 = 10.149407.$
5. $f = \cos 9^\circ + \sin (90 + 9^\circ) - (\sin 39^\circ + \sin 21^\circ)$
 $= \cos 9^\circ + \cos 9^\circ - 2 \sin 30^\circ \cos 9^\circ$
 $= \cos 9^\circ = \cos (9'' \times \frac{1}{2}) = \cos 10'' = \sin 90''.$
6. Let x = reqd. base; then, if n be any logarithm to base 10,
 $10^n = x^{2n}, \therefore x^2 = 10, \therefore x = \sqrt{10} = 3.162.$
7. $9.68472 = L \tan x$ $\frac{1}{11}$ of $60'' = \frac{10.5''}{8} = 13''$,
 $9.68465 = L \tan 25^\circ 49'$ $x = 25^\circ 49' 13''.$
- 7
8. $\frac{1}{2} (\log 2 + \log \sec A) = -\log 5 + \frac{1}{2} (\log \cos^2 B - \log \sin C),$
 $\therefore 3 \log 2 + 3 (L \sec A - 10) = -6 \log 5 + 2 [2 (L \cos B - 10) - (L \sin C - 10)],$
 $\therefore 3 \log 2 + 3 L \sec A = 4 L \cos B - 2 L \sin C + 10 - 6 \log 5.$

XIII.

1. While A is between 0 and 180° , $3A$ must be between 0 and 540° ; and $\sin 3A = \frac{1}{2} = \sin 30^\circ = \sin 150^\circ$; also $\sin 30^\circ = \sin 390^\circ$, and $\sin 150^\circ = \sin 510^\circ$, $\therefore A = 10^\circ, 50^\circ, 130^\circ$, or 170° .
2. (1) $f = \sin (\frac{1}{2}\pi - P) + \sin Q = 2 \sin \frac{1}{2} (\frac{1}{2}\pi - P + Q) \cos \frac{1}{2} (\frac{1}{2}\pi - P - Q) = l,$
 (2) $f = \sin (\frac{1}{2}\pi - P) - \sin Q = 2 \cos \frac{1}{2} (\frac{1}{2}\pi - P + Q) \sin \frac{1}{2} (\frac{1}{2}\pi - P - Q) = l.$
3. $-1.8753145 = 2.1246855$ $\cdot 1246998$
 $\cdot 1246672$ $\cdot 1246672$
 $\underline{183}$ $\underline{326}$
- $\frac{123}{128} = \frac{3}{8}$ nearly = .556 nearly, \therefore reqd. no. = .013325556.
4. $l = 2 \cos A \sin B \sin (A+B) + 2 \sin A \cos B \sin (A+B)$
 $- 2 \sin A \sin B \cos (A+B)$
 $= \sin A [\sin (A+B) \cos B - \cos (A+B) \sin B]$
 $+ \sin B [\sin (A+B) \cos A - \cos (A+B) \sin A]$
 $+ \sin (A+B) (\sin A \cos B + \cos A \sin B)$
 $= \sin A \cdot \sin A + \sin B \cdot \sin B + \sin (A+B) \cdot \sin (A+B) = f.$
5. Diff. for $60'' = .61972 - .61936 = 36$; $\frac{3}{8}$ of 36 = 31,
 $\therefore L \cot 67^\circ 23' 52'' = 9.61972 - 31 = 9.61941,$
 $\therefore \log \cot 67^\circ 23' 52'' = 1.61941.$
6. $l = \frac{1}{2} \left\{ \left(\frac{\sin B}{\sin A} + \frac{\sin C}{\sin A} - \frac{\sin A}{\sin A} \right) \left(\frac{\sin C}{\sin B} + \frac{\sin A}{\sin B} - \frac{\sin B}{\sin B} \right) \right\}$
 $= \frac{1}{2} \left\{ \left(\frac{b}{a} + \frac{c}{a} - \frac{a}{a} \right) \left(\frac{c}{b} + \frac{a}{b} - \frac{b}{b} \right) \right\} = \frac{1}{2} \left(\frac{b+c-a}{a} \cdot \frac{c+a-b}{b} \right)$
 $= \frac{2(s-a) \cdot 2(s-b)}{4ab} = \frac{(s-a)(s-b)}{ab} = f.$
7. $\cdot 78329$ $\cdot 78329$
 $\cdot 78313$ $\cdot 78321$ $\frac{1}{16}$ of $60'' = 30''$, $\therefore x = 52^\circ 37' 30''.$
 $\underline{16}$ $\underline{8}$

8. Suppose the \angle s are A and B , then a, b are the sides, 1.7403627
 $b = a \frac{\sin B}{\sin A}$, $\therefore \log b = \log a + L \sin B - L \sin A$, 9.9764927
 $= \log 79.063$, 11.7168554
 $\therefore b = 79.063$, 9.8188779
 1.8979775

XIV.

1. Expression $= 1 + \cos 2\theta + \cos^2 2\theta - (1 + \cos 2\theta) \cos 2\theta = 1$.
 2. In fig. to Art. 58, suppose $PON = 45^\circ - A$, then
 $OPN = 90^\circ - (45^\circ - A) = 45^\circ + A$;
 $\therefore \tan(45^\circ - A) \tan(45^\circ + A) = \frac{PN}{ON} \cdot \frac{ON}{PN} = 1$.
 3. $2f = 2 \sin^2 \frac{1}{2} A - 2 \sin^2 \frac{1}{2} B = 1 - \cos A - (1 - \cos B) = \cos B - \cos A$
 $= \frac{a}{c} - \frac{b}{c} = 2f$.
 4. $A + B > C$, $\therefore \sin \frac{1}{2}(A + B) > \sin \frac{1}{2} C$, or
 $\sin \frac{1}{2} A \cos \frac{1}{2} B + \cos \frac{1}{2} A \sin \frac{1}{2} B > \sin \frac{1}{2} C$; but $\cos \frac{1}{2} B, \cos \frac{1}{2} A$ are proper fractions, \therefore *a fortiori*, $\sin \frac{1}{2} A + \sin \frac{1}{2} B > \sin \frac{1}{2} C$.
 5. $f = \cos^2 \alpha + \cos^2 \beta - [\cos(\alpha + \beta) + \cos(\alpha - \beta)] \cos \omega$
 $= \cos^2 \alpha + \cos^2 \beta - \cos(\alpha - \beta) \cos(\alpha + \beta) - \cos^2 \omega$
 $= \cos^2 \alpha + \cos^2 \beta - (\cos^2 \alpha - \sin^2 \beta) - \cos^2 \omega$
 $= 1 - \cos^2 \omega = l$.
 6. 3786924
 3786802 $\frac{1}{122}$ of $10'' = 2''$; \therefore reqd. $\angle = 67^\circ 18' 32''$.
 122
 7. $\cos B = \frac{a}{c}$, $\therefore c = \frac{a}{\cos B}$ 1.83518 $\frac{1}{122} = .58$
 $\therefore \log c = \log a - (L \cos B - 10)$ 1.78342
 $= 2.05176$ 2.05176
 $\therefore c = 112.658$. $.05154$
 22
 8. $\sin A = \frac{a}{c} \sin C = \frac{15}{35} \cdot \frac{\sqrt{3}}{2}$ 569804 569553
 $= \frac{3\sqrt{3}}{14} = \frac{3\frac{1}{2}}{14}$ 569488 569488
 316 66
 $\frac{1}{122}$ of $60'' = 13''$ nearly,
 $\therefore L \sin A = \frac{1}{2} \log 3 - \log 14 + 10$ $A = 21^\circ 47' 13''$.
 $= 10.715681 - 1.146128 = 9.569553$.

XV.

1. $\sin \theta - \operatorname{cosec} \theta = \frac{\sin^2 \theta - 1}{\sin \theta} = -\frac{\cos^2 \theta}{\sin \theta}$, $\cos \theta - \sec \theta = -\frac{\sin^2 \theta}{\cos \theta}$;
 $\therefore f = 4 \frac{\cos^4 \theta}{\sin^2 \theta} \cdot \frac{\sin^4 \theta}{\cos^2 \theta} + (1 + \sin 2\theta)(1 - \sin 2\theta)$
 $= \sin^2 2\theta + 1 - \sin^2 2\theta = 1$.

$$2. f = (\sin^2 \theta + \cos^2 \theta)(\sin^4 \theta - \sin^2 \theta \cos^2 \theta + \cos^4 \theta) + 3 \sin^2 \theta \cos^2 \theta \\ = \sin^4 \theta + 2 \sin^2 \theta \cos^2 \theta + \cos^4 \theta = (\sin^2 \theta + \cos^2 \theta)^2 = 1.$$

$$3. \begin{array}{r} 9359018 \\ 9358894 \\ \hline 124 \end{array} \quad \begin{array}{r} 9358921 \\ 9358894 \\ \hline 27 \end{array} \quad \begin{array}{l} \frac{27}{124} \text{ of } 10'' = 2''.18, \\ A = 59^\circ 37' 42''.18. \end{array}$$

$$4. f = 1 - \tan^2 \alpha \tan^2 \beta + \frac{\tan^2 \alpha - \tan^2 \beta}{\sin(\alpha + \beta) \sin(\alpha - \beta)} \\ = 1 - (\sec^2 \alpha - 1)(\sec^2 \beta - 1) + \frac{\tan^2 \alpha - \tan^2 \beta}{\sin^2 \alpha - \sin^2 \beta} \\ = 1 - \sec^2 \alpha \sec^2 \beta + \sec^2 \alpha + \sec^2 \beta - 1 + \frac{1}{\cos^2 \alpha \cos^2 \beta} = 1.$$

$$5. l = 2 \sin \frac{1}{2}(\alpha + \beta) \left[\cos \frac{1}{2}(\alpha - \beta) - \cos \frac{1}{2}(\alpha + \beta + 2\gamma) \right] \\ = \sin \alpha + \sin \beta - [\sin(\alpha + \beta + \gamma) - \sin \gamma] = f.$$

$$6. \frac{b+c}{a} = \frac{b}{a} + \frac{c}{a} = \cot A + \operatorname{cosec} A = \frac{\cos A + 1}{\sin A} = \frac{2 \cos^2 \frac{1}{2} A}{2 \sin \frac{1}{2} A \cos \frac{1}{2} A} \\ = \frac{\cos \frac{1}{2} A}{\sin \frac{1}{2} A}, \therefore f = l.$$

$$7. a^2 = c^2 - b^2 = (c+b)(c-b) = 89168 \times 23686, \\ \therefore \log a = \frac{1}{2}(\log 89168 + \log 23686) = 4.66235, \therefore a = 45956.$$

$$8. \text{ Complete the parallelogram } AD \text{ on } AB, AC; \text{ join } AD \text{ meeting } \\ BC \text{ in } E, AE \text{ is bisector. Now } ACD = 180^\circ - A, \text{ and } AD = 2AE, \\ \therefore (2AE)^2 = AC^2 + CD^2 - 2AC \cdot CD \cos ACD = b^2 + c^2 + 2bc \cos A, \\ \therefore f = l.$$

$$9. \text{ Suppose } O' \text{ (fig. 32) centre of inscribed circle and } \rho \text{ its radius,} \\ \rho = O'A \sin A = (OA - r - \rho) \sin A = \left(\frac{r}{\sin A} - r - \rho \right) \sin A, \\ \therefore \rho(1 + \sin A) = r(1 - \sin A), \therefore \rho = \frac{1 - \sin A}{1 + \sin A} r.$$

$$10. s = 900, s-a = 400, s-b = 300, s-c = 200, \\ \Delta^2 = 900 \times 400 \times 300 \times 200 = 3^3 \cdot 2^3 \cdot 10^8, \\ 2 \log \Delta = 3 \log 3 + 3 \log 2 + 8 = 10.33445, \therefore \log \Delta = 5.167225, \\ \text{diff. for } 1 = 30, \text{ pro. pt.} = 20.5, \frac{20.5}{30} = .68\bar{3}, \\ \therefore \Delta = 146968.3 \text{ sq. links} = 1 \text{ ac. } 4 \text{ sq. ch. } 6968.3 \text{ sq. links.}$$

XVI.

$$1. \text{ For reqd. circ. me., } 180^\circ : \frac{858^\circ}{100 \times 60 \times 60} = \pi : x = \frac{3.1416 \times 858}{100 \times 60 \times 60 \times 180} \\ = \frac{1309 \times 143}{4500000000} = .0000416 \text{ nearly. If } x = \text{reqd. dist. of sun in m.,} \\ \frac{4000}{x} = .0000416, \therefore x = \frac{4000}{.0000416} = 96,000,000 \text{ about.}$$

$$2. \log \left(\sqrt[14]{\frac{1}{3}} \right)^{107} = \log \left(\frac{1}{3} \right)^{\frac{107}{14}} = \frac{107}{14} (-\log 3) = -6.37297736 = 7.6270226. \\ = \log .00000042366.$$

3. $\therefore A+B=180^\circ-C=90^\circ$, $\therefore \frac{1}{2}(A+B)=45^\circ$, $\therefore \cot \frac{1}{2}(A+B)=1$,
 $\therefore \frac{\cot \frac{1}{2}A \cot \frac{1}{2}B - 1}{\cot \frac{1}{2}A + \cot \frac{1}{2}B} = 1$, $\therefore f = l$.
4. $4f = 2(\cos 40^\circ - \cos 60^\circ) \cos 20^\circ = \cos 60^\circ + \cos 20^\circ - 2 \times \frac{1}{2} \cos 20^\circ = \frac{1}{2}$.
5. $2 \log \tan A = \log .96386 = \overline{1}.9840140$, $\therefore L \tan A = 9.9920070$;
 \therefore pro. part = $20070 - 19143 = 927$; and $\frac{2.27}{1000}$ of $60'' = 22''$,
 $\therefore A = 44^\circ 28' 22''$.
6. $a \left(\frac{\sin A}{\cos A} - \frac{\sin \frac{1}{2}(A+B)}{\cos \frac{1}{2}(A+B)} \right) = b \left(\frac{\sin \frac{1}{2}(A+B)}{\cos \frac{1}{2}(A+B)} - \frac{\sin B}{\cos B} \right)$
 $\therefore a [\sin A \cos \frac{1}{2}(A+B) - \cos A \sin \frac{1}{2}(A+B)] \cos B$
 $= b [\sin \frac{1}{2}(A+B) \cos B - \cos \frac{1}{2}(A+B) \sin B] \cos A$,
 $\therefore a \sin \frac{1}{2}(A-B) \cos B = b \sin \frac{1}{2}(A-B) \cos A$, $\therefore f = l$.
7. Expression = $\frac{1}{\cos A} \sin A + \frac{\tan B \cot C + 1}{\cot C} = \tan A + \tan B + \tan C$,
 which is symmetrical with respect to A , B , and C .
8. $\angle OAE = 90^\circ - C$, $\angle OBD = 90^\circ - C$, $\therefore OA = \frac{OE}{\sin OAE} = \frac{OE}{\cos C}$
 and $OD = OB \sin OBD = BO \cos C$, $\therefore AO \cdot OD = BO \cdot OE$;
 and similarly = $CO \cdot OF$.
9. Let $C=120^\circ$, and $b=4a$. Then $\sin B = \frac{b}{a} \sin A = 4 \sin (120^\circ + B)$
 $= 4 (\frac{1}{2} \sqrt{3} \cos B - \frac{1}{2} \sin B)$, $\therefore \cot B = \frac{1}{3} \sqrt{3}$. Similarly $\cot A = 3 \sqrt{3}$.
10. Constructing as in fig. 34, let $DE=x$, $CD=h$, $DAE=DBE=15^\circ$;
 then a circle will go round $ABED$; $\therefore h(h+x) = ab$;

$$\therefore \tan DAE = \tan (CAE - CAD) = \frac{\frac{h+x}{a} - \frac{h}{a}}{1 + \frac{h(h+x)}{a^2}} = \frac{\frac{x}{a}}{1 + \frac{h}{a}}$$

 $\therefore x = (a+b) \tan 15^\circ = \&c.$

XVII.

1. Let θ = circ. meas. reqd.; then arc = $r\theta$, $\therefore \frac{2r\theta}{r\theta + 2r} = \frac{2\theta}{\theta + 2} = 3 - \sqrt{5}$,
 $\therefore \theta(\sqrt{5}-1) = 6 - 2\sqrt{5} = (\sqrt{5}-1)^2$, $\therefore \theta = \sqrt{5}-1$.
2. $\tan \theta = \frac{\sin \theta}{\sqrt{1-\sin^2 \theta}} = \frac{5}{13} \cdot \frac{1}{\sqrt{1-\frac{25}{169}}} = \frac{5}{13} \cdot \frac{13}{\sqrt{144}} = \frac{5}{12}$;
 $\tan 2\theta = 2 \times \frac{5}{12} \times \frac{1}{1-\frac{25}{169}} = \frac{5}{6} \times \frac{144}{119} = \frac{120}{119}$.
3. Make $xOP = A$ of any magnitude (fig. 35), and $yOP' = -A$; then $xOP = 90^\circ - A$. Making $OP' = OP$, the triangles yOP and xOP' are equal in all respects, having $PN = ON'$ in magnitude. These also have the same sign; for \therefore the angles xOP and yOP' are equal in magnitude, when P is above xO , P' is always to the right of yO , and when below, to the left. Therefore in magnitude and sign $PN = ON'$, $\therefore \frac{PN}{OP} = \frac{ON'}{OP'}$, $\therefore \sin A = \cos (90^\circ - A)$.

4. By Arith. Prog., $4 \sec(\frac{2}{3}\pi + \theta) = 2 \sec \theta + 2 \sec(\frac{4}{3}\pi - \theta)$,
 $\therefore 4 \cos \theta \cos(\frac{2}{3}\pi - \theta) = 2 \cos(\frac{2}{3}\pi + \theta) \cos(\frac{2}{3}\pi - \theta) + 2 \cos(\frac{4}{3}\pi + \theta) \cos \theta$,
 $\therefore 2 \cos \frac{2}{3}\pi + 2 \cos(\frac{2}{3}\pi - 2\theta) = \cos \frac{4}{3}\pi + \cos 2\theta + \cos(\frac{2}{3}\pi + 2\theta) + \cos \frac{2}{3}\pi$,
 $\therefore 2 \cos(\frac{2}{3}\pi - 2\theta) = 2 \cos(\frac{1}{3}\pi + 2\theta) \cos \frac{1}{3}\pi = \cos(\frac{1}{3}\pi + 2\theta)$
 $= -\cos(\frac{2}{3}\pi - 2\theta)$, $\therefore \cos(2\theta - \frac{2}{3}\pi) = 0 = \cos(2n\pi \pm \frac{1}{3}\pi)$,
 $\therefore 2\theta = 2n\pi + \frac{2}{3}\pi \pm \frac{1}{3}\pi$, $\therefore \theta = n\pi + \frac{1}{3}\pi \pm \frac{1}{6}\pi$.
5. $f = \frac{1}{\cos^2 \frac{1}{2}\theta \cos \theta} \cdot \frac{\operatorname{cosec}^2 \frac{1}{2}\theta - \operatorname{cosec}^2 \frac{3}{2}\theta}{\operatorname{cosec}^2 \frac{3}{2}\theta} = \frac{\sin^2 \frac{3}{2}\theta}{\cos^2 \frac{1}{2}\theta \cos \theta} \cdot \frac{\sin^2 \frac{3}{2}\theta - \sin^2 \frac{1}{2}\theta}{\sin^2 \frac{3}{2}\theta \sin^2 \frac{1}{2}\theta}$
 $= \frac{4 \sin 2\theta \sin \theta}{\sin^2 \theta \cos \theta} = \frac{8 \sin^2 \theta \cos \theta}{\sin^2 \theta \cos \theta} = 8$.
6. Circumf. $= \pi \cdot 2r = 3 \cdot 1416 \text{ in.} \times 1\frac{1}{2} = 3 \cdot 1416 \text{ in.} + .7854 \text{ in.} = 3 \cdot 927 \text{ in.}$
7. $\sin A \sin(B - C) = \sin(A - B) \sin C$, $\therefore \sin(B + C) \sin(B - C) = \sin(A + B) \sin(A - B)$, $\therefore \sin^2 B - \sin^2 C = \sin^2 A - \sin^2 B$;
 \therefore proportionally, $b^2 - c^2 = a^2 - b^2$.
8. $\log c = \log a + L \sin C - L \sin A$ 2.03933 11.96350
 $= 2.09513$, 9.92417 9.86837
 $\therefore c = 124.48$. 11.96350 2.09513

9. $f = \tan^{-1} \frac{\frac{a}{b+c} + \frac{b}{a+c}}{1 - \frac{ab}{(b+c)(a+c)}} = \tan^{-1} \frac{a^2 + b^2 + (a+b)c}{(a+b)c + c^2}$
 $= \tan^{-1} \frac{c^2 + (a+b)c}{(a+b)c + c^2} = \tan^{-1} 1 = 45^\circ$.
10. $c^2 = a^2 + b^2 - 2ab \cos C = a^2 + b^2 - 2ab(1 - 2 \sin^2 \frac{1}{2}C)$
 $= (a-b)^2 + 4ab \sin^2 \frac{1}{2}C = (44)^2 + 4 \cdot 169 \cdot 125 \left(\frac{33}{65}\right)^2$
 $= 1936 + 4 \cdot (13)^2 \cdot 5^3 \left(\frac{33}{13 \cdot 5}\right)^2 = 154^2$, $c = 154$.

XVIII.

1. Let $\alpha^\circ =$ reqd. unit, and $A^\circ =$ any \angle ; then $\frac{A}{\alpha} = \angle A^\circ$, measured
in terms of reqd. unit. Hence, by question, $\frac{A}{\alpha} + A = \frac{1}{2}A$;
 $\therefore \frac{1}{\alpha} = \frac{1}{9}$, $\therefore \alpha^\circ = 9^\circ$.
2. $f = 2(\sin^2 \theta + \cos^2 \theta)(\sin^4 \theta - \sin^2 \theta \cos^2 \theta + \cos^4 \theta) - 3 \sin^4 \theta - 3 \cos^4 \theta + 1$
 $= 1 - (\sin^4 \theta + 2 \sin^2 \theta \cos^2 \theta + \cos^4 \theta) = 1 - (\sin^2 \theta + \cos^2 \theta)^2 = 1 - 1 = 0$.
3. $f = \sin^2 A + \cos^2 A + \sin^2 B + \cos^2 B - 2(\cos A \cos B + \sin A \sin B)$
 $- 2[1 - \cos(A - B)] = 1 + 1 - 2 \cos(A - B) - 2 + 2 \cos(A - B) = 0$.
4. Expression $= (-1)^3 (-\tan \frac{1}{2}\pi)^3 (-\tan \frac{1}{2}\pi)^5 [4 \tan^4 \frac{1}{2}\pi + 7(-\tan \frac{1}{2}\pi)]$
 $= -(-\sqrt{3})^3 \left(-\frac{1}{\sqrt{3}}\right)^5 (4-7) = -\left(-\frac{1}{\sqrt{3}}\right)^3 (-3) = 1$.
5. $\sin \theta = \pm 1 = \sin(\pm \frac{1}{2}\pi)$, $\therefore \theta = n\pi \pm \frac{1}{2}\pi$, which are all included
in $n\pi + \frac{1}{2}\pi$.

6. $f = \cos^{-1}(1-a) + \cos^{-1}(1-b)$
 $= \cos^{-1}\{(1-a)(1-b) - \sqrt{[(2a-a^2)(2b-b^2)]}\}$
 $= \text{vers}^{-1}\{1 - (1-a-b+ab) + \sqrt{[ab(2-a)(2-b)]}\} = l.$
7. $\frac{\sin A + \sin B + \sin C}{2 \sin A \sin B \sin C} = \left(\frac{2\Delta}{bc} + \frac{2\Delta}{ca} + \frac{2\Delta}{ab} \right) + 2 \frac{2\Delta \cdot 2\Delta \cdot 2\Delta}{bc \cdot ca \cdot ab}$
 $= \left(\frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab} \right) \times \frac{a^2 b^2 c^2}{8 \Delta^2} = \frac{2s}{abc} \times \frac{a^2 b^2 c^2}{8 \Delta^2} = \frac{abc}{4 \Delta} + \frac{\Delta}{s} = \frac{R}{r}.$
8. $\sin^{-1} \cos(x-a) = \cos^{-1} \sin(x-\beta) = \sin^{-1} \sqrt{1 - \sin^2(x-\beta)}$
 $= \sin^{-1} [\cos(x-\beta)], \therefore \cos(x-a) = \pm \cos(x-\beta),$
 $\therefore \cos^2(x-a) - \cos^2(x-\beta) = 0, \therefore \sin(a-\beta) \sin(2x-a-\beta) = 0,$
 $\therefore \sin(2x-a-\beta) = 0 = \sin n\pi, \therefore x = \frac{1}{2}(n\pi + a + \beta).$
9. In fig. 2, Art. 73, suppose CD drawn bisecting BB' and \therefore perpendicular to it. Then $\therefore A=45^\circ$, $CD=AD=\frac{1}{2}(c+c')$, $BD=\frac{1}{2}(c-c')$,
 $\therefore \cos BCB' = \cos^2 BCD (1 - \tan^2 BCD) = \frac{1 - \tan^2 BCD}{1 + \tan^2 BCD}$
 $= \left\{ 1 - \left(\frac{c-c'}{c+c'} \right)^2 \right\} + \left\{ 1 + \left(\frac{c-c'}{c+c'} \right)^2 \right\} = \frac{4cc'}{(c+c')^2} \times \frac{(c+c')^2}{2(c^2+c'^2)} = \frac{2cc'}{c^2+c'^2}.$
10. Suppose the perpendicular from A upon BC (fig. to Art. 20). Then
 $AD \times a = 2\Delta$, also H. M. between $\frac{\Delta}{s-b}$ and $\frac{\Delta}{s-c}$
 $= 2 \frac{\Delta}{s-b} \cdot \frac{\Delta}{s-c} + \left(\frac{\Delta}{s-b} + \frac{\Delta}{s-c} \right) = \frac{2\Delta}{a} = AD.$

XIX.

1. $\sec^2 \theta - \sec \theta - 2 = 0, \therefore \sec \theta = \frac{1}{2} [1 \pm \sqrt{1+8}] = 2 \text{ or } -1, \therefore \theta = \&c.$
2. $f = 4 \cos \alpha (\cos^2 \alpha - \sin^2 \frac{3}{4}\pi) = 4 \cos^3 \alpha - 4 \cos \alpha \frac{3}{4} = l.$
3. $3 \sin 2\theta + 2 \cos 2\theta = 2 + 1 + \cos 2\theta, \therefore \cos 2\theta = 3(1 - \sin 2\theta),$
 $\therefore \cos^2 \theta - \sin^2 \theta = 3(\cos \theta - \sin \theta)^2,$
 $\therefore (1) \cos \theta - \sin \theta = 0, \therefore \tan \theta = 1 = \tan \frac{1}{4}\pi, \therefore \theta = n\pi + \frac{1}{4}\pi;$
 $(2) \cos \theta + \sin \theta = 3(\cos \theta - \sin \theta), \therefore 4 \sin \theta = 2 \cos \theta, \therefore \tan \theta = \frac{1}{2}.$
4. $f = 2 \sin 54^\circ \cos 15^\circ - 2 \sin 36^\circ \cos 15^\circ = 2 \cos 15^\circ (\sin 54^\circ - \sin 36^\circ)$
 $= 2 \frac{\sqrt{3}+1}{2\sqrt{2}} \cdot 2 \cos 45^\circ \sin 9^\circ = l.$
5. $2 \sin 3A \cos 4A = 0, \therefore \sin 7A - \sin A = 0, \therefore \sin 7A = \sin A$
 $\sin [n \cdot 180^\circ + (-1)^n A], \therefore [7 - (-1)^n] A = n \cdot 180^\circ, \therefore A = \text{Ans.}$
6. $n \cdot 180^\circ + (-1)^n 34^\circ$. Let n be even and $= 2m$; the expression becomes $2m \cdot 180^\circ + 34^\circ$, and represents angles having the same tangent as 34° , for they are bounded by the same lines as 34° . Let n be odd and $= 2m+1$; then the expression is $2m \cdot 180^\circ + 180^\circ - 34^\circ$, which represents angles in the second quadrant, which therefore have *not* the same tangent.
7. $2\pi r = 2m \cdot 407 \text{ yds.} = 3927 \text{ yds.},$
 $\therefore r = \frac{3927}{2 \times 3.1416} \text{ yds.} = \frac{39270000}{62832} \text{ yds.} = 625 \text{ yds.}$

8. $\sin(\pi \cos \theta) = \sin(2n\pi + \frac{1}{2}\pi)$, $\therefore \pi \cos \theta = (2n + \frac{1}{2})\pi$,
 $\therefore \cos \theta = 2n + \frac{1}{2} = \pm \frac{1}{2}$ (only possible values) $= \cos \frac{1}{2}\pi$ or $\cos(\pi + \frac{1}{2}\pi)$,
 $\therefore \theta = 2n\pi + \frac{1}{2}\pi$ or $2n\pi + (\pi + \frac{1}{2}\pi) = n\pi + \frac{1}{2}\pi$.

9. Expression $= \tan^{-1} \frac{x \cos \theta}{1 - x \sin \theta} - \tan^{-1} \frac{x - \sin \theta}{\cos \theta}$
 $= \tan^{-1} \frac{\frac{x \cos \theta}{1 - x \sin \theta} - \frac{x - \sin \theta}{\cos \theta}}{1 + \frac{x \cos \theta (x - \sin \theta)}{\cos \theta (1 - x \sin \theta)}} = \tan^{-1} \frac{\sin \theta (1 - 2x \sin \theta + x^2)}{\cos \theta (1 - 2x \sin \theta + x^2)}$
 $= \tan^{-1} \tan \theta = \theta$.

10. $\frac{1 - \sin 2\theta}{1 + \sin 2\theta} = \frac{(1 - \sin \theta)^2}{1 - \sin^2 \theta} = \frac{1 - \sin \theta}{1 + \sin \theta}$, $\therefore \sin 2\theta = \sin \theta$,

$\therefore 2\theta = n\pi + (-1)^n \theta$, $\therefore \theta = \frac{n}{2 - (-1)^n} \pi$.

Equating the cancelled factor to 0, $\sin \theta = 1$, $\therefore \theta = 2n\pi + \frac{1}{2}\pi$.

11. Let x ft. = reqd. height; then $2 - \sqrt{3} = \tan 15^\circ = \left(\frac{x+10}{50} - \frac{x}{50} \right)$

$+ \left(1 + \frac{x(x+10)}{50 \cdot 50} \right) = \frac{10}{50} + \frac{2500 + x^2 + 10x}{2500} = \frac{500}{2500 + x^2 + 10x}$;

solving, x = imaginary quantity. The impossibility of the problem also follows from $\tan 15^\circ$ being $> \frac{1}{2}$.

12. Let r be radius, then $\alpha r = 2r + \beta r$, $\therefore \alpha - \beta = 2$.

XX.

- $\tan \theta (\tan \theta + \sqrt{3}) = 0$, $\therefore (1) \tan \theta = 0$, $\therefore \theta = n\pi$;
 (2) $\tan \theta = -\sqrt{3}$, $\therefore \theta = n\pi + \frac{2}{3}\pi$.
- In fig. 36, $AE \cdot CD \cdot BF = c \cos A \cdot b \cos C \cdot a \cos B = AF \cdot BD \cdot CE$,
 by symmetry; otherwise $= b \cos A \cdot c \cos B \cdot a \cos C = AF \cdot BD \cdot CE$.
- $f = 2 \sin 3A (\cos 2A - \cos 12A) = \sin 5A + \sin A - (\sin 15A - \sin 9A) = l$.
- $\tan^2 \theta - 2 \tan \theta + 1 = \tan \theta + 4 \sqrt{(\tan \theta) + 4}$, $\therefore \tan \theta - 1 = \sqrt{(\tan \theta) + 4}$;
 $\therefore \tan \theta - \sqrt{(\tan \theta) + 4} = 0$, $\therefore \sqrt{(\tan \theta) + 4} = \frac{1}{2}(1 \pm \sqrt{13})$,
 $\therefore \tan \theta = \frac{1}{2}(14 \pm 2\sqrt{13}) = \frac{1}{2}(7 \pm \sqrt{13})$.
- Let $\alpha - 2\beta$, $\alpha - \beta$, α , $\alpha + \beta$, $\alpha + 2\beta$ be the angles in A. P. Then
 $\cos(\alpha - 2\beta) + \cos(\alpha - \beta) + \cos \alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) = 0$,
 $\therefore 2 \cos \alpha \cos 2\beta + 2 \cos \alpha \cos \beta = 0$, $\therefore \cos 2\beta + \cos \beta = 0$,
 $\therefore 2 \cos \frac{3}{2}\beta \cos \frac{1}{2}\beta = 0$, $\therefore \beta = \frac{2}{3}(n\pi + \frac{1}{2}\pi)$, $2(n\pi + \frac{1}{2}\pi)$.
- The figure is easy; in it $BD = c \frac{\sin \frac{1}{2}A}{\sin(\frac{1}{2}A + C)} = a \frac{\sin C}{\sin A} \frac{\sin \frac{1}{2}A}{\sin(\frac{1}{2}A + C)}$
 $= \frac{a \sin C}{2 \cos \frac{1}{2}A \sin(\frac{1}{2}A + C)} = \frac{a \sin C}{\sin(A + C) + \sin C} = \frac{a \sin C}{\sin B + \sin C}$
 Similarly for CD .
- $\sin(\sin \theta) = \frac{1}{2} = \sin \frac{1}{2}\pi$, $\therefore \sin \theta = n\pi + (-1)^n \frac{1}{2}\pi$,
 $\therefore \theta = \sin^{-1} [n\pi + (-1)^n \frac{1}{2}\pi]$.

8. When $\sin 2\theta$ is +ve, 2θ is between $2n\pi$ and $2n\pi + \pi$; $\therefore \theta$ is between $n\pi$ and $n\pi + \frac{1}{2}\pi$; and between these limits $\sin \theta$ and $\cos \theta$ are of like sign, and \therefore their product +ve. When $\sin 2\theta$ is -ve, 2θ is between $2n\pi + \pi$ and $2n\pi + 2\pi$; $\therefore \theta$ is between $n\pi + \frac{1}{2}\pi$ and $n\pi + \pi$; between these limits $\sin \theta$ and $\cos \theta$ are opposite in sign, and \therefore their product -ve.
9. $\tan^{-1} x + \cot^{-1} x = n\pi + \tan^{-1} x + n\pi + \cot^{-1} x$
 $= 2n\pi + \tan^{-1} x + \frac{1}{2}\pi - \tan^{-1} x = 2n\pi + \frac{1}{2}\pi.$
10. Let A, B (fig. 37) be the spectators, MH the vertical to the horizontal plane through A, B . Then $AB = 10$ m., $HBM = 45^\circ$, $\therefore BH = HM = x$ suppose; $HAM = 30^\circ$, $\therefore AH = HM \cot 30^\circ = x\sqrt{3}$. Also $BAH = 22\frac{1}{2}^\circ$. Then $x^2 = 10^2 + (\sqrt{3}x)^2 - 20\sqrt{3}x \cos 22\frac{1}{2}^\circ$;
 $\therefore 2x^2 - 10\sqrt{6} + 3\sqrt{2}x + 100 = 0$,
 $\therefore x = \frac{10\sqrt{6} + 3\sqrt{2} \pm \sqrt{[100(6 + 3\sqrt{2}) - 800]}}{4}$
 $= \frac{1}{2}[\sqrt{6 + 3\sqrt{2}} \pm \sqrt{3\sqrt{2} - 2}] = \frac{1}{2}[3.2 \pm 1.5]$ nearly $= 11\frac{1}{2}$ or $4\frac{1}{2}$.
11. Let p, P be perimeters of the figures. Then (Euc. iv. 15) radius of circle about hexagon $= \frac{1}{2}p$; and radius of circle in triangle
 $\left(-\frac{\Delta}{s}\right) = \frac{1}{2} \cdot \frac{P}{3} \cdot \frac{P}{3} \cdot \frac{\sqrt{3}}{2} \times \frac{2}{P} = \frac{P}{6\sqrt{3}}$, $\therefore \pi \frac{36}{p^2} : \pi \frac{P^2}{36.3} = 3 : 4$,
 $\therefore p^2 : P^2 = 1 : 4$, $\therefore p : P = 1 : 2$.
12. In fig. 38, suppose A, B, C the harbour, blockader, and ship when overtaken, respectively. Then $\angle B = 45^\circ - 23^\circ = 22^\circ$, $AB = 10$, $BC = 15$;
 $\therefore \tan \frac{1}{2}(A - C) = \frac{a - c}{a + c} \cot \frac{1}{2}B$ $\frac{3821}{1294} \quad \frac{821}{777}$
 $= \frac{1}{3} \cot 11^\circ = \frac{1}{3} \cot 11^\circ$;
 $\therefore L \tan \frac{1}{2}(A - C) = \log 2 + L \cot 11^\circ - 1$ $\frac{2527}{1143}, \text{ of } 60'' = 1''.$
 $= 10.0123777$;
 $\therefore \frac{1}{2}(A - C) = 45^\circ 48' 59''$, and $\frac{1}{2}(A + C) = 79^\circ$;
 adding, $A = 124^\circ 48' 59''$, \therefore course reqd. is $10^\circ 11' 1''$ S. of E.

XXI.

1. $\tan A + \cot A = \frac{\sin A}{\cos A} + \frac{\cos A}{\sin A} = \frac{\sin^2 A + \cos^2 A}{\sin A \cos A} = \frac{2}{\sin 2A}$;
 and $\therefore \sin 2A$ is never > 1 , $\therefore \tan A + \cot A$ is never < 2 .
2. Let A be reqd. \angle . Then $L \tan A = L \cot A + .5316768$
 $= 10 - \log \tan A + .5316768 = 20 - L \tan A + .5316768$,
 $\therefore L \tan A = 10.2658384$, $\therefore A = 61^\circ 32'$.
3. $(2bc \cos A)^2 = (b^2 + c^2 - a^2)^2 = b^4 + c^4 + a^4 + 2b^2c^2 - 2a^2b^2 - 2a^2c^2$,
 $\therefore 1 - \cos^2 A = 1 - \frac{b^4 + c^4 + a^4 + 2b^2c^2 - 2a^2b^2 - 2a^2c^2}{4b^2c^2}$
 $= \frac{4b^2c^2 - b^4 - c^4 - a^4 - 2b^2c^2 + 2a^2b^2 + 2a^2c^2}{4b^2c^2}$, $\therefore f = l$.

$$4. \quad 2 \cos^2 \frac{1}{2} A = 1 + \cos A = 1 + \frac{b^2 + c^2 - a^2}{2bc} = \frac{(b+c)^2 - a^2}{2bc},$$

$$\text{and } 2 \sin^2 \frac{1}{2} A = 1 - \cos A = 1 - \frac{b^2 + c^2 - a^2}{2bc} = \frac{a^2 - (b-c)^2}{2bc},$$

$\therefore f = l$. When $a = b = c$, $A = 60^\circ$ and $\cot^2 \frac{1}{2} A = 3$;

$$\text{also } \frac{(b+c)^2 - a^2}{a^2 - (b-c)^2} = \frac{(2a)^2 - a^2}{a^2 - 0} = \frac{3a^2}{a^2} = 3.$$

5. Let a, b be any two chords in the same circle, respectively subtending at the circumference the angles α and β . Then, by Art. 138, if R be radius of the circle, $a = 2R \sin \alpha$, $b = 2R \sin \beta$,
 $\therefore a : b = \sin \alpha : \sin \beta$.

$$6. \quad \tan \theta = \sec \phi - \tan \phi = \frac{1 - \sin \phi}{\cos \phi} = \frac{(\cos \frac{1}{2} \phi - \sin \frac{1}{2} \phi)^2}{\cos^2 \frac{1}{2} \phi - \sin^2 \frac{1}{2} \phi} = \frac{\cos \frac{1}{2} \phi - \sin \frac{1}{2} \phi}{\cos \frac{1}{2} \phi + \sin \frac{1}{2} \phi}$$

$$= \frac{1 - \tan \frac{1}{2} \phi}{1 + \tan \frac{1}{2} \phi} = \frac{\tan \frac{1}{2} \pi - \tan \frac{1}{2} \phi}{1 + \tan \frac{1}{2} \pi \tan \frac{1}{2} \phi} = \tan (\frac{1}{2} \pi - \frac{1}{2} \phi),$$

$$\therefore \theta = \pi\pi + \frac{1}{2}\pi - \frac{1}{2}\phi.$$

7. $\sin 112^\circ$ is $> \cos 112^\circ$ and +ve; hence

$$\sin 112^\circ + \cos 112^\circ = \sqrt{(1 + \sin 224^\circ)} = \sqrt{(1 + .69)} = \sqrt{1.69} = 1.3,$$

$$\sin 112^\circ - \cos 112^\circ = \sqrt{(1 - \sin 224^\circ)} = \sqrt{(1 - .69)} = \sqrt{1.69} = 1.3,$$

$$\therefore 2 \sin 112^\circ = 1.8568, \text{ and } \sin 112^\circ = .9284.$$

$$8. \quad \text{Let } \theta = \tan^{-1} \frac{x \sin \theta}{1 + x \cos \theta}, \therefore \sec^2 \theta = 1 + \left(\frac{x \sin \theta}{1 + x \cos \theta} \right)^2$$

$$= \frac{1 + 2x \cos \theta + x^2}{1 + 2x \cos \theta + x^2 \cos^2 \theta}, \therefore f = \cos 2\theta = 2 \cos^2 \theta - 1 =$$

$$\frac{2(1 + 2x \cos \theta + x^2 \cos^2 \theta)}{1 + 2x \cos \theta + x^2} - 1 = \frac{1 + 2x \cos \theta + x^2(2 \cos^2 \theta - 1)}{1 + 2x \cos \theta + x^2} = l.$$

9. In fig. 39, let $AB = 40$ ft. $B = 90^\circ$, $A = 75^\circ$, $AO = OC = 25$. Then since B is in a semicircle, being right, $\therefore AC$ is a diameter; $\therefore ADC$ is right-angled; $\therefore AC = 50$ ft., $AB = 40$ ft., $\therefore BC = 30$ ft.,
 $\therefore \triangle ABC = \frac{1}{2} \times 40 \times 30$ sq. ft. = 600 sq. ft.; and if $CAB = \alpha$,

$$\therefore \sin 2CAD = \sin (150^\circ - 2\alpha) = \frac{1}{2} (2 \cos^2 \alpha - 1) + \frac{1}{2} \sqrt{3} \cdot 2 \sin \alpha \cos \alpha$$

$$= \frac{1}{2} - \frac{1}{2} + \sqrt{3} \cdot \frac{1}{2} \cdot \frac{4}{5} = \frac{1}{5} (7 + 24\sqrt{3}),$$

$$\therefore \triangle ADC = \frac{1}{2} AD \cdot DC = \frac{1}{2} CA^2 \sin CAD \cdot \cos CAD = \frac{1}{2} (CA^2) \sin 2CAD$$

$$= \frac{1}{2} (50^2) \cdot \frac{1}{5} (7 + 24\sqrt{3}) \text{ sq. ft.} = \frac{50^2}{2} (7 + 24\sqrt{3}) \text{ sq. ft.},$$

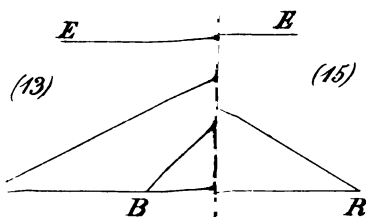
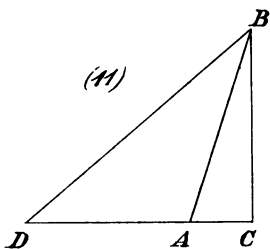
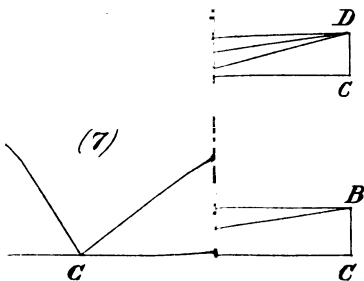
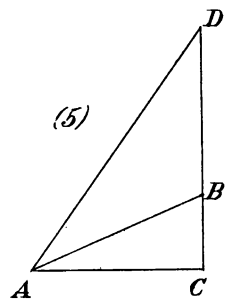
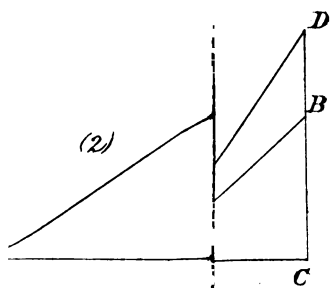
$$\therefore \text{area } ABCD = (600 + 87\frac{1}{2} + 300\sqrt{3}) \text{ sq. ft.}$$

$$= (687.5 + 519.6) \text{ sq. ft.} = 1207.1 \text{ sq. ft.}$$

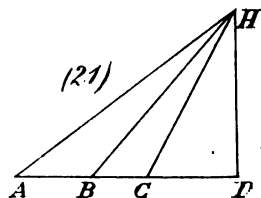
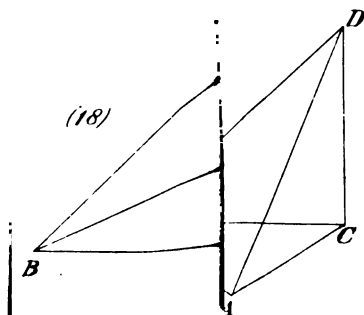
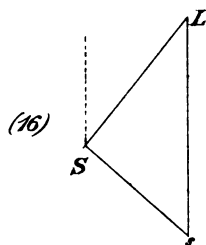
10. Let a be base. Now $s(s-a)(s-b)(s-c) = s(s-a)[s^2 - (b+c)s + bc]$, where $s, s-a$ are constant; and \therefore also $2s-a$ or $b+c$. Hence bc alone is variable. Now $4bc = (b+c)^2 - (b-c)^2$, which is great when $b = c$; i.e., when the triangle is isosceles.

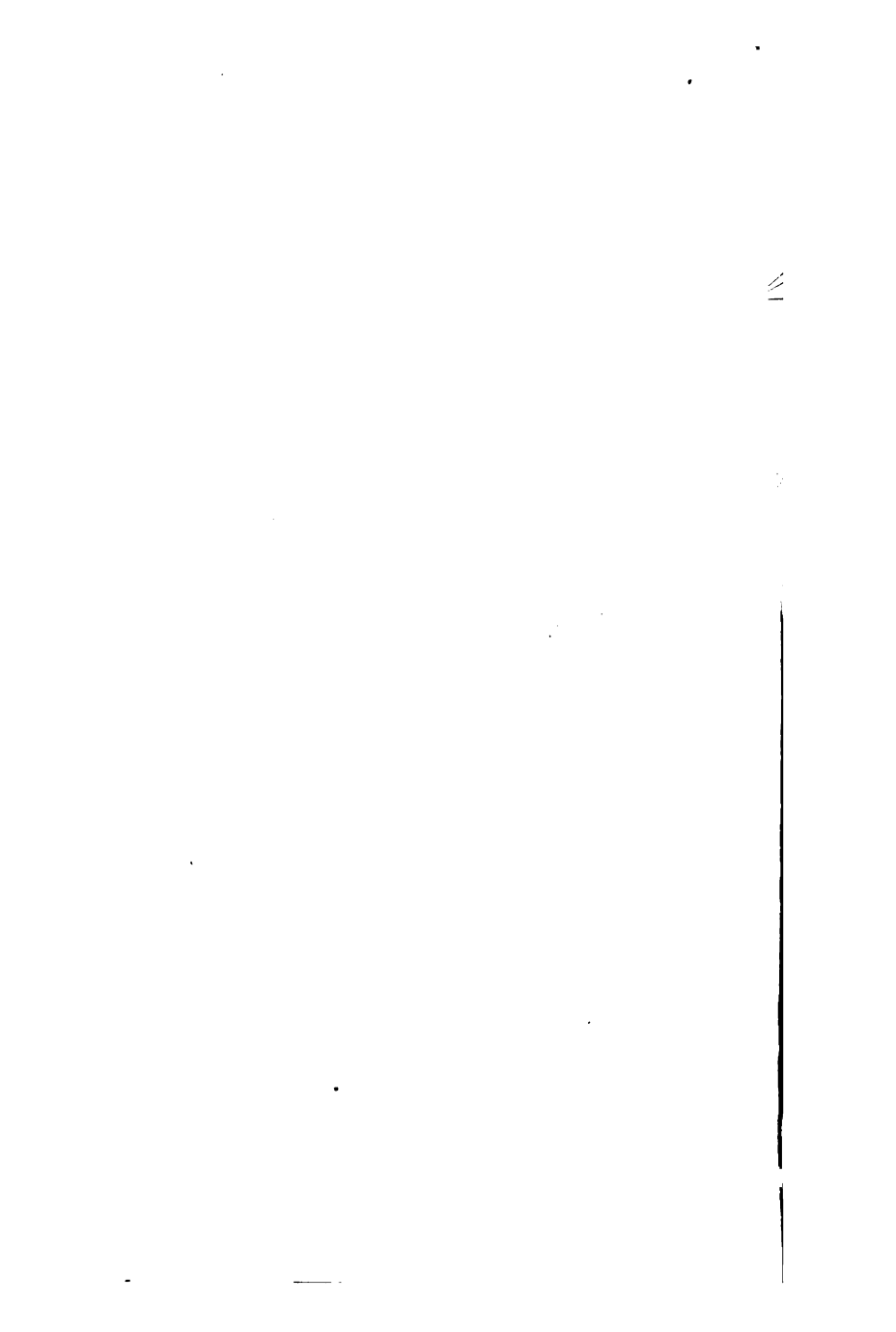
11. In fig. 40, let D, E be respective points of contact of inscribed and escribed circles with BC . Then $CD \sim CE = OD \cot \frac{1}{2} C \sim O'E \tan \frac{1}{2} C$
 $= \frac{\Delta}{s} \sqrt{\frac{s(s-c)}{(s-a)(s-b)}} \sim \frac{\Delta}{s-a} \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} = (s-c) \sim (s-b) = b \sim c.$

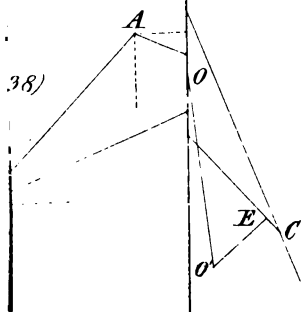
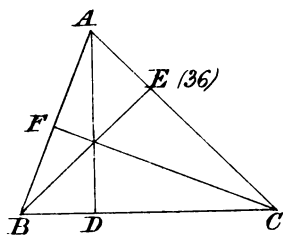
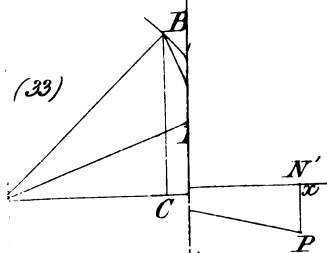
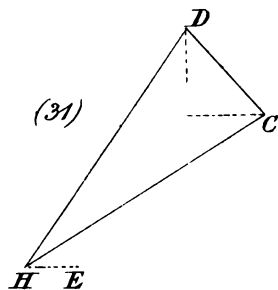
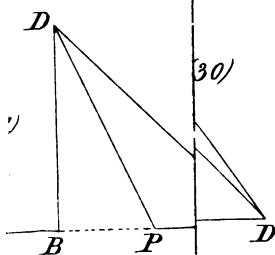
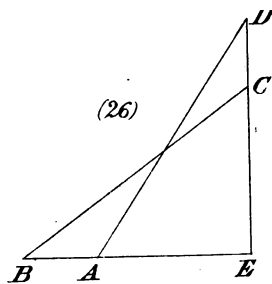
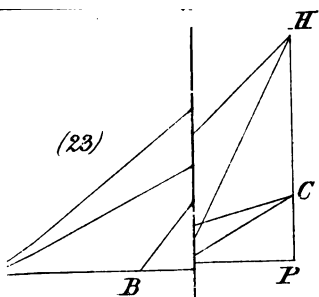
12. (1) \because the elevations are all equal, the points of observation are equally distant from the foot of the tower. This latter therefore is the centre of the circle passing through those points. Hence distance = radius of circle about the triangle formed by the given distances, and may be known by formula $r = \frac{\Delta}{s}$. The required height is perpendicular in a right-angled triangle whose base and base-angle are known.



(15)







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